Weak law of large numbers does not apply to Cauchy distribution

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This report solves Exercise 8.55 from the textbook, *Probability, an introduction.*

Exercise 8.55 Let X_1, X_2, ... be i.i.d. Cauchy-distributed random variables. Show that $A_n = n^{-1} \sum_{i=1}^{n} X_i$ converges in distribution to the Cauchy distribution as $n \to \infty$. Compare this with the conclusion of the weak law of large numbers.

Solution

According to Wikipedia, for a random variable X obeying the Cauchy distribution with the location parameter $x_0 \in \mathbb{R}$ and the scale parameter $\gamma \in \mathbb{R}_+$,

- its mean and variance are not undefined;
- its characteristic function is $\exp(x_0 it - \gamma|t|)$.

According to page 68 of the lecture notes, the characteristic function has the following properties.

1. If a, $b \in \mathbb{R}$, $\phi_{aX+b}(t) = \exp(itb)\phi_x(at)$.
2. If X and Y are independent, $\phi_{X+Y} = \phi_X \phi_Y$.

By the results above,

$$\begin{aligned}
\phi_{nA_n}(t) &= \phi_{X_1}(t)\phi_{X_2}(t)...\phi_{X_n}(t) \\
&= \exp(\sum_{j=1}^{n} x_0 it - \gamma|t|) \\
&= \exp(nx_0 it - n\gamma|t|). \\
\phi_{A_n}(t) &= \phi_{nA_n}(\frac{t}{n}) \\
&= \exp(nx_0 i\frac{t}{n} - n\gamma|\frac{t}{n}|) \\
&= \exp(x_0 it - \gamma|t|). \\
\lim_{n\to\infty} \phi_{A_n}(t) &= \exp(x_0 it - \gamma|t|). \qquad (1)
\end{aligned}$$

By the uniqueness theorem, the LHS of (1) is the characteristic function of the Cauchy distribution with the location parameter as x_0 and the scale parameter as γ.

Denote by A a random variable obeying this distribution.

Then, as $n \to \infty$, A_n converges in distrubution to A, which also obeys the Cauchy distribution.

Q.E.D.

According to page 137 of the textbook,

Theorem 8.17 (Weak law of large numbers) Let X_1, X_2, ... be a sequence of independent random variables, each with mean μ and variance σ^2. The average of the first n of the X_i satisfies, as $n \to \infty$,

$$n^{-1} \sum_{i=1}^{n} X_i \to \mu, \text{ in probability.}$$

It might be tempting to apply to this exercise the weak law of large numbers due to the form of A_n, taking x_0 as μ. However, the truth is that the mean and the variance are undefined for the Cauchy distribution. Therefore, the weak law of large numbers does not apply to this exercise.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- `https://en.wikipedia.org/wiki/Cauchy_distribution`