4 Theoretical Exercises on Poisson Processes

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This report solves several exercises from the textbook, *Probability, an introduction.* It includes

- finding the formulae for $\mathbb{P}(N_t = 0)$ (Exercise 11.3), $\mathbb{P}(N_t \text{ is even})$, and $\mathbb{P}(N_t \text{ is odd})$ (Exercise 11.19);
- proving $\lim_{t\to\infty} \text{var}(N_t/t) = 0$ (Exercise 11.18)
- discussing the central limit theorem for a Poisson process (Exercise 11.20).

Exercise 11.3 If N is a Poisson process with rate λ, show that

$$\mathbb{P}(N_{t+h} = 0) = [1 - \lambda h + o(h)]\mathbb{P}(N_t = 0). \tag{1}$$

for small positive values of h. Hence, show that $p(t) = \mathbb{P}(N_t = 0)$ satisfies the differential equation

$$p'(t) = -\lambda p(t). \tag{2}$$

Solve this equation to find $p(t)$.

Solution
According to page 100 of the lecture notes,

$$\mathbb{P}(N_{t+h} = n | N_t = n) = 1 - \lambda h + o(h).$$

Therefore,

$$\begin{aligned}
\mathbb{P}(N_{t+h} = 0 | N_t = 0) &= 1 - \lambda h + o(h) \\
\frac{\mathbb{P}(N_{t+h} = 0 \cap N_t = 0)}{\mathbb{P}(N_t = 0)} &= 1 - \lambda h + o(h) \\
\frac{\mathbb{P}(N_{t+h} = 0)}{\mathbb{P}(N_t = 0)} &= 1 - \lambda h + o(h), \text{ since } N_{t+h} = 0 \text{ implies } N_t = 0; \\
\mathbb{P}(N_{t+h} = 0) &= [1 - \lambda h + o(h)]\mathbb{P}(N_t = 0).
\end{aligned}$$

(1) is proved.

$$\begin{aligned}
p'(t) &= \lim_{h\to 0} \frac{p(t+h) - p(t)}{h} \\
&= \lim_{h\to 0} \frac{\mathbb{P}(N_{t+h} = 0) - \mathbb{P}(N_t = 0)}{h} \\
&= \lim_{h\to 0} \frac{[-\lambda h + o(h)]\mathbb{P}(N_t = 0)}{h}, \text{ according to (1);} \\
&= [-\lambda + \lim_{h\to 0} \frac{o(h)}{h}]\mathbb{P}(N_t = 0) \\
&= [-\lambda + \lim_{h\to 0} \frac{o(h)}{h}]\mathbb{P}(N_t = 0) \\
&= (-\lambda + 0)\mathbb{P}(N_t = 0) \\
&= -\lambda p.
\end{aligned}$$

(2) is proved.
(2) can be rewritten as

$$\begin{aligned}
\frac{dp}{dt}(t) &= -\lambda p \\
\int \frac{1}{p}\frac{dp}{dt}(t)dt &= -\int \lambda dt \\
\ln[p(t)] &= -\lambda t + \text{const} \\
p(t) &= c\exp(-\lambda t), \qquad (3)
\end{aligned}$$

where c is a constant.
By (3), $p(0) = c$.
According to page 100 of the lecture notes, $N_0 = 0$.
Therefore,

$$c = p(0) = \mathbb{P}(N_0 = 0) = 1,$$

that is, $c = 1$.
Thus, (3) becomes

$$p(t) = \exp(-\lambda t).$$

That concludes this solution.

Exercise 11.18 If N is a Poisson process with rate λ, show that $\text{var}(N_t/t) \to 0$ as $t \to \infty$.

Solution
According to (11.8) on page 184 of the textbook, $\text{var}(N_t) = \lambda t$.
Therefore,

$$\begin{aligned}
\lim_{t\to\infty} \text{var}(N_t/t) &= \lim_{t\to\infty} \frac{1}{t^2}\text{var}(N_t) \\
&= \lim_{t\to\infty} \frac{1}{t^2}\lambda t \\
&= \lim_{t\to\infty} \frac{\lambda}{t} \\
&= 0.
\end{aligned}$$

Q.E.D.

Exercise 11.19 If N is a Poisson process with rate λ, show that for $t > 0$,

$$\begin{aligned}
\mathbb{P}(N_t \text{ is even}) &= \exp(-\lambda t)\cosh(\lambda t). \\
\mathbb{P}(N_t \text{ is odd}) &= \exp(-\lambda t)\sinh(\lambda t).
\end{aligned}$$

Solution
To solve this exercise, Theorem 11.6 on page 184 of the textbook will be used.

> For each $t > 0$, the random variable N_t has the Poisson distribution with parameter λt. That is, for $t > 0$ and $k = 0, 1, 2, ...,$
>
> $$\mathbb{P}(N_t = k) = \frac{1}{k!}(\lambda t)^k \exp(-\lambda t). \qquad (4)$$

Also, the Taylor series for $sinh(x)$ and $cosh(x)$ will be used.

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{5}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{6}$$

The two series are convergent for all $x \in \mathbb{R}$.

By (4) and (6)

$$\begin{aligned}\mathbb{P}(N_t \text{ is even}) &= \sum_{n=0}^{\infty} \mathbb{P}(N_t = 2n) \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(\lambda t)^{2n} \exp(-\lambda t) \\ &= \cosh(\lambda t)\exp(-\lambda t).\end{aligned}$$

By (4) and (5),

$$\begin{aligned}\mathbb{P}(N_t \text{ is odd}) &= \sum_{n=0}^{\infty} \mathbb{P}(N_t = 2n+1) \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\lambda t)^{2n+1} \exp(-\lambda t) \\ &= \sinh(\lambda t)\exp(-\lambda t).\end{aligned}$$

Q.E.D.

Exercise 11.20 If N is a Poisson process with rate λ, show that the moment generating function of

$$U_t = \frac{N_t - \mathbb{E}(N_t)}{\sqrt{\operatorname{var}(N_t)}} \tag{7}$$

is

$$M_t(x) = \mathbb{E}[\exp(xU_t)] = \exp\{-x\sqrt{\lambda t} + \lambda t[\exp(\frac{x}{\sqrt{\lambda t}} - 1]\}.$$

Deduce that, as $t \to \infty$, for $u \in \mathbb{R}$,

$$\mathbb{P}(U_t \le u) \to \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}v^2)dv.$$

Solution

According to Theorem 11.6 (see page 2 of this report), $N_t \sim P(\lambda t)$.
According to *Moment generating function of Poisson distribution* by Lin Guozhang, for the Poisson distribution with parameter λ, its moment generating function is

$$M(x) = \exp[\lambda(e^x - 1)].$$

By these two results,

$$M_{N_t}(x) = \exp[\lambda t(e^x - 1)]. \tag{8}$$

According to *Properties of the moment generating function* by Lin Guozhang,

$$M_{aY+b}(x) = \exp(xb)M_Y(ax). \tag{9}$$

Also, according to (11.8) on page 184 of the textbook,

$$\mathbb{E}(N_t) = \lambda t, \text{ and } \mathrm{var}(N_t) = \lambda t.$$

By this result and (7),

$$\begin{aligned} U_t &= \frac{N_t - \lambda t}{\sqrt{\lambda t}} \\ &= \frac{1}{\sqrt{\lambda t}} N_t - \sqrt{\lambda t}. \end{aligned} \tag{10}$$

By (8) to (10),

$$\begin{aligned} M_{U_t}(x) &= \exp(-x\sqrt{\lambda t}) \exp[\lambda t(e^{x/\sqrt{\lambda t}} - 1)] \\ &= \exp[-x\sqrt{\lambda t} + \lambda t(e^{x/\sqrt{\lambda t}} - 1)]. \end{aligned} \tag{11}$$

The Taylor series of the exponential function is

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} = 1 + y + \frac{1}{2}y^2 + O(y^3).$$

Thus,

$$e^{x/\sqrt{\lambda t}} = 1 + x/\sqrt{\lambda t} + \frac{1}{2}x^2/(\lambda t) + O(t^{-3/2}). \tag{12}$$

By (11) and (12),

$$\begin{aligned} \lim_{t\to\infty} M_{U_t}(x) &= \lim_{t\to\infty} \exp\{-x\sqrt{\lambda t} + \lambda t[1 + x/\sqrt{\lambda t} + \frac{1}{2}x^2/(\lambda t) + O(t^{-3/2}) - 1]\} \\ &= \lim_{t\to\infty} \exp\{-x\sqrt{\lambda t} + \lambda t[x/\sqrt{\lambda t} + \frac{1}{2}x^2/(\lambda t) + O(t^{-3/2})]\} \\ &= \lim_{t\to\infty} \exp[-x\sqrt{\lambda t} + x\sqrt{\lambda t} + \frac{1}{2}x^2 + \lambda t \cdot O(t^{-3/2})] \\ &= \lim_{t\to\infty} \exp[\frac{1}{2}x^2 + O(\frac{1}{\sqrt{t}})] \\ &= \exp[\frac{1}{2}x^2 + \lim_{t\to\infty} O(\frac{1}{\sqrt{t}})] \\ &= \exp(\frac{1}{2}x^2) \end{aligned}$$

By the continuity theorem, as $t \to \infty$, for $u \in \mathbb{R}$,

$$\mathbb{P}(U_t \le u) \to \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}v^2) dv.$$

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- https://en.wikipedia.org/wiki/Hyperbolic_functions
- *Moment generating function of Poisson distribution* by Lin Guozhang
- *Properties of the moment generating function* by Lin Guozhang