Asymmetric random walk revisits it starting point only finitely often with probability 1

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This report solves Exercise 10.10 of the textbook, *Probability, an introduction.*

Exercise 10.10 Show that $u_n = \mathbb{P}(S_n = S_0)$ satisfies

$$\sum_{n=0}^{\infty} u_n \begin{cases} < \infty & \text{if } p \neq q, \\ = \infty & \text{if } p = q, \end{cases} \tag{1}$$

and deduce that an asymmetric random walk revisits it starting point only finitely often with probability 1. Stirling's formula is needed for this exercise, which is the following.

$$n! \sim (\frac{n}{e})^n \sqrt{2\pi n} \text{ as } n \to \infty, \tag{2}$$

where $\sim$ means that the two quantities are asymptotic: their ratio tends to 1 as n tends to infinity.

Solution

This exercise is related to testing convergence of the infinite series.
In the book of *Thomas' Calculus* by George Thomas, two test are introduced.

1. Limit comparison test.

   Suppose that $a_n > 0$ and $b_n > 0$ for all $n > N \in \mathbb{N}$.

   (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
   (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
   (c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

2. Root test.
   Let $\sum_{n=0}^{\infty} a_n$ be any series.

$$\text{If } \lim_{n\to\infty} (|a_n|)^{1/n} \begin{cases} < 1, & \text{the series converges;} \\ = 1, & \text{the test is inconclusive;} \\ > 1, & \text{the series diverges.} \end{cases}$$

The two tests will be used to solve this exercise.
The theorem on page 89 of the lecture notes will also be used.

Let $u_n := \mathbb{P}(S_n = S_0)$, that is, the probability that the particle revisits its starting point at time n. Then, if n is odd, $u_n = 0$. If $n = 2m$ is even, $u_{2m} = \binom{2m}{m} p^m q^m$.

According to the theorem,

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= \sum_{m=0}^{\infty} u_{2m} \\ &= \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m \\ &:= \sum_{m=0}^{\infty} a_m, \text{ where } a_m := \binom{2m}{m} p^m q^m. \end{aligned} \tag{3}$$

Perform the limit comparison test for the case $p = q = \frac{1}{2}$.
As a p-series with $p' = \frac{1}{2}$, $\sum_{m=0}^{\infty} b_m = \infty$, where $b_m = \frac{1}{\sqrt{m}}$. (p' is primed in order to distinguish from p, the probability of the particle moving one unit rightwards during each unit of time.)

$$\begin{aligned}\lim_{m\to\infty} \frac{a_m}{b_m} &= \lim_{m\to\infty} \binom{2m}{m} p^m q^m \sqrt{m} \\ &= \lim_{m\to\infty} \frac{2m!}{m!m!} p^m q^m \sqrt{m} \\ &= \lim_{m\to\infty} \frac{(\frac{2m}{e})^{2m}\sqrt{2\pi 2m}}{(\frac{m}{e})^{2m}(2\pi m)} p^m q^m \sqrt{m}, \text{ according to (2);} \\ &= \sqrt{\frac{1}{\pi}} \lim_{m\to\infty} 2^{2m} \sqrt{\frac{1}{m}} p^m q^m \sqrt{m} \\ &= \sqrt{\frac{1}{\pi}} \lim_{m\to\infty} 2^{2m} p^m q^m \\ &= \sqrt{\frac{1}{\pi}} \lim_{m\to\infty} 2^{2m} (\frac{1}{2})^m (\frac{1}{2})^m \\ &= \sqrt{\frac{1}{\pi}} \lim_{m\to\infty} 1 \\ &= \sqrt{\frac{1}{\pi}}. \end{aligned}$$

This corresponds to situation (a) (see the previous page).
Since $\sum_{m=0}^{\infty} b_m = \infty$, $\sum_{m=0}^{\infty} a_m = \infty$.
Therefore, by (3), for $p = q = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} u_n = \sum_{m=0}^{\infty} a_m = \infty. \tag{4}$$

Now, we consider the case where $p \neq q$. It is useful to recall the inequality of arithmetic and geometric means before performing the test.

$$a + b \geq 2\sqrt{ab}, \tag{5}$$

where a and b are nonnegative numbers, and the equals sign holds if and only if $a = b$.

By (5),

$$\begin{aligned} p + q &\geq 2\sqrt{pq} \\ 1 &\geq 2\sqrt{pq} \\ pq &\leq \frac{1}{4}. \end{aligned}$$

Since we are now considering the case where $p \neq q$, we have

$$pq < \frac{1}{4}. \tag{6}$$

Perform the root test for the case $p \neq q$. (Some calculation results above when performing the limit comparison test will be used in the following.)

$$\begin{aligned}
\lim_{m\to\infty} (|a_m|)^{1/m} &= \lim_{m\to\infty} (2^{2m}\sqrt{\frac{1}{\pi m}} p^m q^m)^{1/m} \\
&= \lim_{m\to\infty} (2^{2m}\sqrt{\frac{1}{\pi m}} p^m q^m)^{1/m} \\
&= 4pq \lim_{m\to\infty} (\frac{1}{\pi m})^{1/2m} \\
&= 4pq \lim_{m\to\infty} \exp[\ln(\frac{1}{\pi m})^{1/2m}] \\
&= 4pq \lim_{m\to\infty} \exp[\frac{1}{2m}\ln(\frac{1}{\pi m})] \\
&= 4pq \exp[\lim_{m\to\infty} \frac{1}{2m}\ln(\frac{1}{\pi m})] \\
&= 4pq \exp(\lim_{m\to\infty} -\frac{1}{2m}), \text{ where L'Hôspital's Rule is used;} \\
&= 4pq \exp(0) \\
&= 4pq \\
&< 1, \text{ which is according to (6).}
\end{aligned}$$

Therefore, for $p \neq q$,

$$\sum_{n=0}^{\infty} u_n = \sum_{m=0}^{\infty} a_m < \infty. \tag{7}$$

By (4) and (7), (1) is proved.

For an asymmetric random walk, $p \neq q$.
Therefore,

$$\sum_{n=0}^{\infty} u_n < \infty.$$

To deduce that an asymmetric random walk revisits its starting point only finitely often with probability 1, we use the following lemma.

First Borel-Cantelli lemma:
Let $\{A_n\}$ be a sequence of events such that $\sum_{n=0}^{\infty} \mathbb{P}(A_n) < \infty$. Then, almost surely, only finitely many A_n's will occur.

Since u_n is the probability that the particle revisits its starting point at time n, by (7) and the lemma, the particle revisits its starting point only finitely often with probability 1. (In probability theory, an event is said to happen almost surely if it happens with probability 1.)

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.

- *Thomas' Calculus* by George Thomas
- https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means
- https://nptel.ac.in/content/storage2/courses/108106083/lecture10_bclemma.pdf
- https://en.wikipedia.org/wiki/Almost_surely
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