Branching processes for $p_k = pq^k$

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This report solves Exercise 9.17 and 9.23 of the textbook, *Probability, an introduction.*

Exercise 9.17 Find the mean and variance of Z_n when the family-size distribution is given by $p_k = pq^k$ for k =0, 1, 2, ..., and 0< $p = 1 - q < 1$. Deduce that $\text{var}(Z_n) \to 0$ if and only if $p > \frac{1}{2}$.
Solution
Mean:
According to page 83 of the lecture notes,

$$\mathbb{E}(Z_n) = \mu^n, \text{ where } \mu := \sum_k kp_k. \tag{1}$$

Therefore, to find the mean, one needs to find μ.
By the definition of p_k given in the question,

$$\mu = \sum_{k=0}^{\infty} kpq^k.$$

By the property of the infinite series,

$$\begin{aligned}
\mu &= \sum_{k=1}^{\infty}(k-1)pq^{k-1} \\
&= \sum_{k=1}^{\infty} kpq^{k-1} - \sum_{k=1}^{\infty} pq^{k-1} \\
&= \frac{1}{q}\sum_{k=1}^{\infty} kpq^k - p\sum_{k=1}^{\infty} q^{k-1} \\
&= \frac{1}{q}\sum_{k=0}^{\infty} kpq^k - p\sum_{k=0}^{\infty} q^k \\
&= \frac{1}{q}\mu - p\sum_{k=0}^{\infty} q^k.
\end{aligned}$$

By the formula for the infinite sum of the geometric series,

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}. \tag{2}$$

Therefore,

$$\begin{aligned}
\mu &= \frac{pq}{(1-q)^2} \\
&= \frac{p(1-p)}{p^2} \\
&= \frac{1-p}{p}.
\end{aligned} \tag{3}$$

By (1) and (3)

$$\begin{aligned}\mathbb{E}(Z_n) &= \mu^n \\ &= (\frac{1-p}{p})^n.\end{aligned}$$

Variance:
As proved in *Deriving the variance for the size of the n-th generation of the branching process* by Nguyen Duc Thanh and *Variance of branching process* by Peng Qi,

$$\operatorname{var}(Z_n) = \begin{cases} n\sigma^2 & \text{if } \mu = 1, \\ \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1, \end{cases} \tag{4}$$

where σ^2 is the variance of the family-size distribution.
Therefore, to find the variance, one needs to find σ^2.

$$\begin{aligned}\sigma^2 &= \sum_{k=0}^{\infty} k^2 pq^k - \mu^2 \\ &= \sum_{k=1}^{\infty} (k-1)^2 pq^{k-1} - \mu^2 \\ &= \sum_{k=1}^{\infty} k^2 pq^{k-1} - 2\sum_{k=1}^{\infty} kpq^{k-1} + \sum_{k=1}^{\infty} pq^{k-1} - \mu^2 \\ &= \frac{1}{q}\sum_{k=1}^{\infty} k^2 pq^k - 2\sum_{k=1}^{\infty} kpq^{k-1} + \sum_{k=1}^{\infty} pq^{k-1} - \mu^2 \\ &= \frac{1}{q}\sum_{k=0}^{\infty} k^2 pq^k - 2\sum_{k=0}^{\infty} (k+1)pq^k + \sum_{k=0}^{\infty} pq^k - \mu^2 \\ &= \frac{1}{q}\sum_{k=0}^{\infty} k^2 pq^k - 2\sum_{k=0}^{\infty} kpq^k - p\sum_{k=0}^{\infty} q^k - \mu^2 \\ &= \frac{1}{q}(\sigma^2 + \mu^2) - 2\mu - p\sum_{k=0}^{\infty} q^k - \mu^2. \end{aligned} \tag{5}$$

By (2), (3), and (5),

$$\sigma^2 = \frac{1-p}{p^2}. \tag{6}$$

Thus, the variance can be calculated according to (3), (4), and (6).

The following deduces $\operatorname{var}(Z_n) \to 0$ if and only if $p > \frac{1}{2}$.
Proof
Consider 3 cases.
Case 1: $p = \frac{1}{2}$.
By (3), $\mu = 1$. By (6), $\sigma^2 = 2$.
According to (4),

$$\lim_{n\to\infty} \operatorname{var}(Z_n) = \lim_{n\to\infty} 2n = \infty.$$

Case 2: $p < \frac{1}{2}$.
By (3), $\mu > 1$.
Therefore,

$$\lim_{n\to\infty} \mu^{n-1} = \infty.$$

$$\lim_{n\to\infty} \mu^n = \infty.$$

According to (4),

$$\lim_{n\to\infty} \mathrm{var}(Z_n) = \frac{\sigma^2}{\mu - 1} \lim_{n\to\infty} \mu^{n-1}(\mu^n - 1) = \infty.$$

Case 3: $p > \frac{1}{2}$.
By (3), $0 < \mu < 1$.

$$\lim_{n\to\infty} \mu^{n-1} = 0.$$

$$\lim_{n\to\infty} \mu^{n} = 0.$$

According to (4),

$$\lim_{n\to\infty} \mathrm{var}(Z_n) = 0.$$

Case 1 and 2 have proved the "only if" direction, since $0 < p \le \frac{1}{2}$ implies $\lim_{n\to\infty} \mathrm{var}(Z_n) \neq 0$.
Case 3 has proved the "if" direction.

Q.E.D.

Exercise 9.23 If the family-size distribution of a branching process has mass function $p_k = pq^k$ for k=0, 1, 2, ... and $0 < p = 1 - q < 1$, use Theorem 9.19 to show that the probability that the process becomes extinct ultimately is $\frac{p}{q}$ if $p \le \frac{1}{2}$.

Solution

Theorem 9.19 states that the probability e of ultimate extinction is the smallest non-negative root of the equation

$$x = G(x). \tag{7}$$

By the definition of the probability generating function,

$$G(x) = \sum_{k=0}^{\infty} x^k p_k.$$

According to the p_k given in the question,

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} x^k pq^k \\ &= \frac{p}{1 - xq}, \text{ for } |x| < \frac{1}{q}. \end{aligned}$$

Assume a solution exists for $0 < x < \frac{1}{q}$ to (7). Then,

$$x = \frac{p}{1 - xq}$$

$$qx^2 - x + p = 0$$

$$qx^2 - x + (1 - q) = 0$$

$$(x - 1)[qx - (1 - q)] = 0$$

$$x_1 = 1,\ x_2 = \frac{1 - q}{q} = \frac{p}{q}.$$

If $0 < p \le \frac{1}{2}$, $x_2 \le x_1$. According to the theorem, $e = x_2 = \frac{p}{q}$.

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- *Deriving the variance for the size of the n-th generation of the branching process* by Nguyen Duc Thanh
- *Variance of branching process* by Peng Qi,