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5. *Lack-of-memory property.* If X has the geometric distribution with parameter p, show that

$$\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)$$

for $m, n = 0, 1, 2, \dots$.

We say that X has the 'lack-of-memory property' since, if we are given that $X - m > 0$, then the distribution of $X - m$ is the same as the original distribution of X. Show that the geometric distribution is the only distribution concentrated on the positive integers with the lack-of-memory property.

*1 Show that $\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)$

We know that for geometric distribution,

$$\mathbb{P}(X = k) = pq^{k-1} \quad \text{for } k = 1, 2, 3, \dots \text{ and } q = 1 - p \qquad (2.16)$$

thus,

$$\mathbb{P}(X > k) = \mathbb{P}(X = k+1) + \mathbb{P}(X = k+2) + \mathbb{P}(X = k+3) + \dots$$

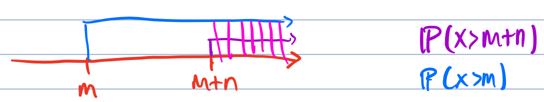
$$= pq^{k} + pq^{k+1} + pq^{k+2} + \dots$$

$$= pq^{k}\underbrace{(1 + q + q^2 + q^3 + \dots)}_{\text{as this goes to } \infty}$$

$$= pq^{k}\left(\frac{1}{1-q}\right) = pq^{k}\left(\frac{1}{p}\right) = q^{k} \qquad (1)$$

$$P(X > m+n \mid X > m) = \frac{\mathbb{P}(X > m+n \cap X > m)}{\mathbb{P}(X > m)}$$

assuming that m and n are non-negative integers, $m+n > m$, and the intersection between $m+n$ and m is simply $m+n$.



$$= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)}$$

substituting this into the probability mass function,

$$= \frac{q^{m+n}}{q^{m}} = q^{n} = \mathbb{P}(X > n) \quad \square$$

#2 Show that lack of memory property only applies to the geometric distribution (for positive integers)

Lack of memory property shows that

$$\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)$$

$$\Leftrightarrow \mathbb{P}(X > m+n \cap X > m) = \mathbb{P}(X > n)\,\mathbb{P}(X > m)$$

$$\Leftrightarrow \mathbb{P}(X > m+n) = \mathbb{P}(X > m)\,\mathbb{P}(X > n)$$

for $m, n > 0$ observe that

$$\mathbb{P}(X > 2) = \mathbb{P}(X > 1)\,\mathbb{P}(X > 1) = [\mathbb{P}(X > 1)]^2$$

$$\mathbb{P}(X > 3) = \mathbb{P}(X > 1)\,\mathbb{P}(X > 2) = \mathbb{P}(X > 1)\cdot[\mathbb{P}(X > 1)]^2 = [\mathbb{P}(X > 1)]^3$$

$$\mathbb{P}(X > 4) = \mathbb{P}(X > 1)\,\mathbb{P}(X > 3) = \mathbb{P}(X > 1)\cdot[\mathbb{P}(X > 1)]^3 = [\mathbb{P}(X > 1)]^4$$

$\Rightarrow$ Thus, we can clearly see that $\mathbb{P}(X > k) = [\mathbb{P}(X > 1)]^k$

Observe as well: $\mathbb{P}(X > 1) = 1 - \mathbb{P}(X \leq 1)$

$$= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1)$$

$$= 1 - 0 - \mathbb{P}(X = 1)$$

$$= 1 - \mathbb{P}(X = 1)$$

Therefore, $\mathbb{P}(X > k) = [1 - \mathbb{P}(X = 1)]^k$ let $p = \mathbb{P}(X = 1)$

thus $q = 1 - \mathbb{P}(X = 1)$

$\Leftrightarrow \mathbb{P}(X > k) = q^k$ $\square$ equivalent with 1 above which was derived from the probability mass function of the geometric distribution.

$\Rightarrow$ Hence, probability with geometric distribution implies lack of memory and lack of memory property implies that the distribution is geometric.