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Special Mathematics Lecture
Introduction to Probability (Spring 2021)
Report: The random walk in $\mathbb{Z}^3$ is transient

Theorem: The simple, symmetric walk on $\mathbb{Z}^3$ is transient.

Proof:

+ Suppose we start at the origin.

We can only return to origin after an even number $2n$ steps, since it requires i steps leftward and i steps rightward, j steps upward and j steps downward, k steps forward and k step backward to return to origin, given that

$$i+j+k=n$$

+ Thus, the probability that the particle return to origin in $2n$ steps is:

$$p_{00}^{(2n)} = \sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} \underbrace{\left(\frac{1}{6}\right)^{2n}}_{\text{symmetric walk in } \mathbb{Z}^3} \cdot \underbrace{\frac{(2n)!}{(i!\,j!\,k!)^2}}_{\text{number of ways it can be done}}$$

+ Note that $\dfrac{(2n)!}{(i!\,j!\,k!)^2} = \dfrac{(2n)!}{n!\,n!} \cdot \left(\dfrac{n!}{i!\,j!\,k!}\right)^2 = \dbinom{2n}{n}\left(\dfrac{n!}{i!\,j!\,k!}\right)^2$

$$\rightsquigarrow p_{00}^{(2n)} = \sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \left(\frac{n!}{i!\,j!\,k!}\right)^2 \left(\frac{1}{3}\right)^{2n} \quad (*)$$

+ Further, we have:

$$\begin{cases} \displaystyle\sum_{\substack{i+j+k=n \\ i,j,k}} \left(\frac{1}{3}\right)^n \cdot \frac{n!}{i!\,j!\,k!} = \left(3\cdot\frac{1}{3}\right)^n = 1 \\ \left(\dfrac{1}{3}\right)^n \dfrac{n!}{i!\,j!\,k!} < 1 \end{cases}$$

Using the inequality: $\sum_i a_i^2 \leq \max\{a_i\}$ given that $\sum_i a_i = 1$ and $a_i \geq 0$

we have: $\displaystyle\sum_{i+j+k=n} \left(\frac{n!}{i!\,j!\,k!}\cdot\frac{1}{3^n}\right)^2 \leq \max\left\{\frac{1}{3^n}\,\frac{n!}{i!\,j!\,k!}\right\} = M$

Thus, from (*), we have.

$$p_{0,0}^{(2n)} \leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \max\left\{ \frac{1}{3^n} \frac{n!}{i!\,j!\,k!} : i+j+k=n;\ i,j,k \geq 0 \right\}$$

$$\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \quad .$$

+ Applying Stirling formula for RHS:

$$\Rightarrow p_{00}^{(2n)} \leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sim \left(\frac{1}{2}\right)^{2n} \cdot \frac{(2n)^{2n}\sqrt{2n}}{(n^n\sqrt{n})^2} \cdot M \sim \frac{1}{\sqrt{n}}$$

$$\text{where } M \sim \frac{1}{3^n} \left.\frac{n^n\sqrt{n}}{i^i\sqrt{i}\,j^j\sqrt{j}\,k^k\sqrt{k}}\right|_{i\,j\,k \text{ maximize } M}$$

with

$$\frac{1}{3^n} \left.\frac{n^n\sqrt{n}}{i^i\sqrt{i}\,j^j\sqrt{j}\,k^k\sqrt{k}}\right|_{i\,j\,k \text{ maximize } M} \leq \frac{1}{3^n} \frac{n^n\sqrt{n}}{\left[\left(\frac{n}{3}\right)^{n/3}\sqrt{\frac{n}{3}}\right]^3}$$ by the method of Lagrange multiplier

$$\leq (\sqrt{3})^3 \cdot \frac{1}{n}$$

$$\Rightarrow p_{00}^{(2n)} \leq C \cdot \frac{1}{\sqrt{n}} \cdot \frac{1}{n} \leq \frac{C}{n^{3/2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(2n)} \leq \sum_{n=0}^{\infty} \frac{C}{n^{3/2}} < \infty$$

↑ by integral test for convergence

as $p_{00}^{(2n+1)} = 0$

$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(n)} < \infty \Rightarrow \text{the walk is transient}$$ ∎