Dam Truyen Duc
NUID: 061801876

Special Mathematics Lecture
Introduction to Probability (Spring 2021)
Report: Moment generating function for some distributions.

I present the calculation for explicit form of the moment generating function for a discrete distribution and some continuous distributions

① Geometric distribution:

$$P(X=j) = (1-p)^{j-1}p = q^{j-1}p \qquad (q = 1-p)$$

⇝ Moment generating function

$$M_X(t) = E(e^{tX})$$
$$= \sum_{j=1}^{\infty} e^{tj}(q^{j-1}p)$$
$$= p.q^{-1}\sum_{j=1}^{\infty}(e^t q)^j$$
$$= \frac{p}{q} \cdot \frac{e^t q}{1-e^t q}$$

$$\Rightarrow \boxed{M_X(t) = \frac{pe^t}{1-e^t q}}$$

② Exponential distribution with parameter λ:

The pdf : $f_X(x) = \lambda e^{-\lambda x}$ for $x \in [0; \infty)$

⇝ Moment generating function:

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$$
$$= \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx$$
$$= \lambda\int_0^{\infty} e^{(t-\lambda)x}\,dx$$
$$= \lambda \frac{e^{(t-\lambda)x}}{t-\lambda}\Big|_0^{\infty}$$

as $\lim_{x\to\infty} \frac{e^{(t-\lambda)x}}{t-\lambda} = 0$ exists for only $t < \lambda$

⇝ Moment generating function exists for $t<\lambda$ with:

$$M_X(t) = \lim_{x\to\infty}\frac{e^{(t-\lambda)x}}{t-\lambda} - \frac{\lambda}{t-\lambda} = 0 - \frac{\lambda}{t-\lambda}$$

$$\rightsquigarrow \boxed{M_X(t) = +\frac{\lambda}{\lambda - t} \qquad (t < \lambda)}$$

③ Standard normal distribution.

The pdf is : $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

+ We perform Taylor expansion for e^{tx} about $(tx=0)$:
$$e^{tx} = \sum_{m=0}^{\infty} \frac{1}{m!} (tx)^m.$$

$\rightsquigarrow$ Moment generating function:
$$M_X(t) = \int_{-\infty}^{+\infty} f_X(x) e^{tx} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \left(\sum_{m=0}^{\infty} \frac{1}{m!} t^m x^m\right) dx$$

$$\rightsquigarrow M_X(t) = \sum_{m=0}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{m!} t^m \underbrace{\int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx}_{= I_m} \qquad (0)$$

+ We evaluate $I_m = \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx$

- If $m = 2n+1$ with $n \in \mathbb{N}$, then $(x^m e^{-x^2/2})$ is an odd function of x
$$\rightsquigarrow I_{2n+1} = 0 \qquad (1)$$

- If $m = 2n$ (even) $(n \in \mathbb{N})$.
Recall Gaussian integral: $\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$
The Feynman's integral trick allows that:
$$\int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2} dx = \left(-\frac{\partial}{\partial \alpha}\right)^n \left(\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx\right) = \left(-\frac{\partial}{\partial \alpha}\right)^n \frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{(2n-1)!!}{2^n} \frac{\sqrt{\pi}}{(\alpha)^{(2n+1)/2}}$$

$$\rightsquigarrow I_{2n} = \int_{-\infty}^{+\infty} e^{-x^2/2} x^{2n} dx = (2n-1)!! \cdot \sqrt{\pi} \cdot \frac{2^n \sqrt{2}}{2^n} = \frac{(2n)!}{n!\, 2^n} \sqrt{2\pi} \qquad (2)$$

+ Substitute (1) and (2) to (0)

$$\Rightarrow M_X(t) = \sum_{m \text{ even}} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{m!} t^m I_m$$
$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(2n)!} t^{2n} \cdot \frac{(2n)!}{n!\, 2^n} \sqrt{2\pi}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{t^2}{2}\right)^n$$

$$\Rightarrow \boxed{M_X(t) = e^{t^2/2}}$$

$$\leadsto \boxed{M_X(t) = +\frac{\lambda}{\lambda - t} \quad (t < \lambda)}$$

③ Standard normal distribution.

+ The pdf is : $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

+ We can find $M_X(t)$ explicitly as

$$M_X(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{tx} \, dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + tx} \, dx$$

where $-\frac{1}{2}x^2 + tx = -\frac{1}{2}(x^2 - 2tx + t^2) + \frac{t^2}{2}$

$$\leadsto M_X(t) = \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-t)^2} \, d(x-t)$$

$$= \frac{1}{\sqrt{2\pi}} e^{t^2/2} \sqrt{2\pi}$$

$$\boxed{M_X(t) = e^{t^2/2} \quad \boxtimes}$$