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Special Mathematics Lecture
Introduction to Probability (Spring 2021)
Report: Some useful identities

Let $g(x)$ be a function with $-\infty < E(g(x)) < +\infty$ and $-\infty < g(-1) < \infty$

① If X is a random variable having Poisson distribution with parameter λ, then

$$E[\lambda g(X)] = E[X g(X-1)]$$

∇ + The probability mass function of X is

$$f_X(x) = \lambda^x \frac{e^{-\lambda}}{x!}$$

+ Then, we have

$$E[\lambda g(X)] = \sum_{x=0}^{\infty} \lambda g(x) . \lambda^x \frac{e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \lambda^{x+1} g(x) e^{-\lambda} . \frac{(x+1)}{(x+1)!}$$

$$= \sum_{x=0}^{\infty} (x+1) g(x) \left(\frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!} \right)$$

$$= \sum_{y=1}^{\infty} y . g(y-1) \left(\frac{e^{-\lambda} \lambda^{y}}{y!} \right) \quad \text{where } y = x+1$$

$$= 0 . g(-1) . \left(\frac{e^{-\lambda} \lambda^0}{0!} \right) + \sum_{y=1}^{\infty} y g(y-1) \left(\frac{e^{-\lambda} \lambda^y}{y!} \right)$$

$$= \sum_{y=0}^{\infty} (y g(y-1)) \left(\frac{e^{-\lambda} \lambda^y}{y!} \right) \qquad (1)$$

+ On the other hand,

$$E[X g(X-1)] = \sum_{x=0}^{\infty} (x g(x-1)) \left(\frac{e^{-\lambda} \lambda^x}{x!} \right) \qquad (2)$$

+ By (1) and (2) $\Rightarrow$ $E[\lambda g(X)] = E[X g(X-1)]$ ∎

(2) If X is a random variable having negative binomial distribution with parameter $(r; p)$, then

$$E[(1-p)\,g(X)] = E\left[\frac{X}{r+X-1}\, g(X-1)\right]$$

+ The probability mass function of X is

$$f_X(x) = \binom{x+r-1}{r-1}(1-p)^x p^r$$

+ We calculate the expectation value of $g(x)$

$$E[g(x)] = \sum_{x=0}^{\infty} g(x)\binom{x+r-1}{r-1} p^r (1-p)^x$$

$$= \sum_{y=1}^{\infty} g(y-1)\binom{y+r-2}{r-1} p^r (1-p)^{y-1} \quad \text{where } y = x+1$$

$$= \sum_{y=1}^{\infty} g(y-1)\binom{y+r-2}{y-1} p^r (1-p)^{y-1} \quad \text{as } \binom{n}{k} = \binom{n}{n-k}$$

$$= \sum_{y=1}^{\infty} g(y-1) \cdot \frac{(y+r-2)!}{(y-1)!\,(r-1)!} p^r (1-p)^{y-1}$$

$$= \sum_{y=1}^{\infty} g(y-1) \cdot \frac{y}{y+r-1}\binom{y+r-1}{y} p^r (1-p)^{y-1}$$

$$= \sum_{y=0}^{\infty} g(y-1)\, \frac{y}{y+r-1}\binom{y+r-1}{y} p^r (1-p)^{y-1} \quad \text{as summand is 0 at } y=0$$

$$= \sum_{y=0}^{\infty}\left(\frac{y}{y+r-1}\, g(y-1)\right)\left[\binom{y+r-1}{r-1} p^r (1-p)^y\right] \cdot \frac{1}{1-p}$$

But $E\left[\frac{X}{r+X-1}\, g(X-1)\right] = \sum_{x=0}^{\infty}\left(\frac{x}{x+r-1}\, g(x-1)\right)\left(\binom{x+r-1}{r-1} p^r (1-p)^y\right)$

$\Rightarrow$ $E[g(x)] = \frac{1}{1-p}\, E\left[\frac{X}{r+X-1}\, g(X-1)\right]$ where $\frac{1}{1-p}$ is a real number

$\Rightarrow$ $E[(1-p)\, g(X)] = E\left[\frac{X}{r+X-1}\, g(X-1)\right]$ ∎

③ If X is a random variable having gamma distribution with parameters (α,β), then

$$E[g(X)(X-\alpha\beta)] = \beta E[Xg'(X)]$$

+ The pdf of X is
$$f_X(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \quad (x>0)$$

+ We have
$$E[g(x)(x-\alpha\beta)] = \int_0^{+\infty} g(x)(x-\alpha\beta) \cdot \frac{1}{\Gamma(\alpha)} \beta^{-\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Note that $\frac{d}{dx}(x^\alpha e^{-\frac{x}{\beta}}) = e^{-\frac{x}{\beta}}\left(-\frac{x^\alpha}{\beta} + \alpha x^{\alpha-1}\right)$
$$= -(x-\alpha\beta) \cdot e^{-\frac{x}{\beta}} \frac{x^{\alpha-1}}{\beta}$$

$$\Rightarrow E[g(x)(x-\alpha\beta)] = -\int_0^{+\infty} g(x) \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \beta \cdot d(x^\alpha e^{-x/\beta})$$
$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha-1}}\left[-g(x) x^\alpha e^{-x/\beta}\Big|_0^{+\infty} + \int_0^{+\infty} x^\alpha e^{-x/\beta} g'(x)\, dx \right]$$

+ Assume that $g(x)$ is differentiable, $E[Xg'(X)]$ exists and $\lim_{x\to\infty} g(x) x^\alpha e^{-x/\beta} = 0$, then

$$E[g(x)(x-\alpha\beta)] = 0 + \int_0^{+\infty} (g'(x)\cdot x)\, \beta \left(\frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta}\right) dx$$

But $E[X g'(X)] = \int_0^{\infty} (g'(x)\, x) \cdot \left(\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\right) dx$

$$\Rightarrow E[g(X)(X-\alpha\beta)] = \beta E[Xg'(X)] \qquad \blacksquare$$