SML: Introduction to Probability

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Problem 9 Chapter 6 from Grimmett and Welsh's Text

Let X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{1}{4}(x+3y)e^{-(x+y)}, & \text{if } x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal density function of Y,
Show that $\mathbb{P}(Y > X) = 5/8$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} \frac{1}{4}(x+3y)e^{-(x+y)}\,dx$$

$$= \frac{1}{4}\Big[\underbrace{\int_0^{\infty} x e^{-(x+y)}\,dx}_{①} + \int_0^{\infty} 3y e^{-(x+y)}\,dx\Big]$$

$$① = \Big[x(-1)e^{-(x+y)}\Big]_0^{\infty} + \int_0^{\infty} e^{-(x+y)}\,dx$$

$$= -\Big[\underbrace{(xe^{-x})}_{\xrightarrow{x\to\infty} 0} e^{-y}\Big]_0^{\infty} - \Big[e^{-(x+y)}\Big]_0^{\infty}$$

$$= -[0-0] - [0 - e^{-y}]$$

$$= e^{-y}$$

$$\Rightarrow f_Y(y) = e^{-y} + 3y\left[-e^{-(x+y)}\right]_0^{\infty}$$

$$= e^{-y} + 3ye^{-y}$$

$$= e^{-y}(1+3y)$$

$$P(Y>X) = \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy\, f(x,y)$$

$$= \frac{1}{4}\int_0^{\infty} dx \underbrace{\int_x^{\infty} dy\,(x+3y)e^{-(x+y)}}_{2}$$

$$2 = \int_x^{\infty} x e^{-(x+y)} dy + \int_x^{\infty} 3y e^{-(x+y)} dy$$

$$= \left[-xe^{-(x+y)}\right]_x^{\infty}$$

$$+ 3\left[-ye^{-(x+y)}\right]_x^{\infty} - 3\left[e^{-(x+y)}\right]_x^{\infty}$$

$$= \left[0 + xe^{-2x} \right] + 3\left[0 + xe^{-2x} \right]$$

$$- 3\left[-e^{-2x} \right]$$

$$= e^{-2x}\left[4x + 3 \right]$$

$$\Rightarrow P(Y > X) = \frac{1}{4} \int_0^\infty e^{-2x} \left[4x + 3 \right] dx$$

$$= \frac{4}{4} \underbrace{\int_0^\infty \left(x e^{-2x} \right) dx}_{3} + \frac{3}{4} \int_0^\infty e^{-2x} \, dx$$

$$3 = \left[-\frac{1}{2} x e^{-2x} \right]_0^\infty + \int_0^\infty \frac{1}{2} e^{-2x} \, dx$$

$$= -\frac{1}{2} \left[0 - 0 \right] + \frac{1}{2} \left[-\frac{1}{2} e^{-2x} \right]_0^\infty$$

$$= -\frac{1}{4} \left[0 - 1 \right]$$

$$= \frac{1}{4}$$

$$\Rightarrow P(Y > X) = \frac{1}{4} + \frac{3}{4} \int_0^\infty e^{-2x} \, dx$$

$$= \frac{1}{4} + \frac{3}{4}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$= \frac{1}{4} - \frac{3}{8}\left[0 - 1\right]$$

$$= \frac{1}{4} + \frac{3}{8}$$

$$= \frac{5}{8}$$, as required

$\square$