SMC : Introduction to Probability

Dillon Loh Guan Hui (L06180194)

Ex 6.36 from Grimmett and Welsh's Text.

Random Variables X, Y, Z have joint density function

$$f(x,y,z) = \begin{cases} 8xyz, & \text{if } 0<x, y, z<1 \\ 0, & \text{otherwise} \end{cases}$$

Are X, Y and Z independent?
Find $\mathbb{P}(X>Y)$ and $\mathbb{P}(Y>Z)$

$$f_x(x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; f(x,y,z)$$

$$= \int_0^1 dy \int_0^1 dz \; 8xyz$$

$$= 8x \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1$$

$$= 2x$$

By symmetry, it is clear that we also have

$$f_Y(y) = 2y$$

$$f_Z(z) = 2z$$

Since $f(x, y, z) = f_x(x) f_y(y) f_z(z)$

i.e. f can be factorised as the product of 3 functions of x, y, z seperately

$\Rightarrow$ X, Y, Z are independent.

$$P(X > Y) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{y}^{\infty} dx \; f(x, y, z)$$

$$= \int_0^1 dz \int_0^1 dy \int_y^1 dx \; 8xyz$$

$$= \int_0^1 dz \int_0^1 dy \; 8yz \left[\frac{x^2}{2} \right]_y^1$$

$$= \int_0^1 dz \int_0^1 dy \; 4yz \left[1 - y^2 \right]$$

$$= \int_0^1 dz \; 4z \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1$$

$$= \int_0^1 dz \; 4z \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \int_0^1 z \; dz$$

$$= \left[\frac{z^2}{2} \right]_0^1 = \frac{1}{2}$$

$$P(Y > Z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz \int_{z}^{\infty} dy \; f(x, y, z)$$

$$= \int_0^1 dx \int_0^1 dz \int_z^1 dy \; 8xyz$$

$$= \int_0^1 dx \int_0^1 dz \; \frac{8xz}{2} \left[1 - z^2 \right]$$

$$= \int_0^1 dx \; 4x \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \int_0^1 dx \; x$$

$$= \frac{1}{2}$$