SML: Introduction to Probability

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Exercise 3.24 of Grimmett's Probability, an Introduction

Define the indicator function of an event A as I_A, where

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}, \quad \omega \in \Omega.$$

Show that two events A and B are independent (1) iff their indicator functions are independent random variables. (2)

(1) → (2)

If A, B independent, we have

(a) A^c, B^c independent.

Proof: $P(A^c \cap B^c) = 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)P(B)$ 2 from (1)

$= [1 - P(A)][1 - P(B)]$

$= P(A^c) P(B^c)$

$\square$

(b) A, B^C independent

Proof: $P(A \cap B^C) = P(A) - P(A \cap B)$ & ①

$= P(A) - P(A)P(B)$

$= P(A)[1 - P(B)]$

$= P(A)P(B^C)$ ▭

(c) A^C, B independent

Proof: $P(A^C \cap B) = P(B) - P(A \cap B)$

$= P(B)[1 - P(A)]$

$= P(B)P(A^C)$ ▭

Consider $P(I_A = a, I_B = b) = P(I_A(\omega) = a, I_B(\omega) = b)$

$P(I_A = 1, I_B = 1) = P(\omega \in A \cap B) = P(A)P(B) = P(I_A = 1)P(I_B = 1)$

By the same argument above but for

$(I_A, I_B) = (0,0), (0,1), (1,0)$,

we see that P

of ω being in or not in A is independent

from whether ω is in or not in B

$\Rightarrow P(I_A = a, I_B = b) = P(I_A = a)P(I_B = b)$

$\therefore$ ① $\rightarrow$ ② shown, by defn of independent random variable.

(2) $\longrightarrow$ (1)

If I_A, I_B independent, then for $P(A \cap B)$,

$$P(A \cap B) = P(I_A(\omega) = 1, I_B(\omega) = 1)$$

$$= P(I_A = 1) P(I_B = 1)$$ since independent

$$= P(A) P(B)$$

$\Rightarrow$ A & B independent

$\therefore$ (2) $\Rightarrow$ (1) shown.

This ends the proof $\square$.