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Bivariate Normal Distribution.

Univariate normal distribution has density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{for } x \in \mathbb{R}.$$

Similarly, corresponding Bivariate Normal Distribution has a joint density function f as the function of 2 variables x, y:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) \quad \text{for } x, y \in \mathbb{R}.$$

$\rho \in (-1, 1)$.

Normalization condition: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx dy = 1$.

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}}\, e^{c+\frac{b^2}{4a}}}$$

Check: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx dy$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} dx \left[\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dy\right]$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \sqrt{2\pi(1-\rho^2)}\, \exp\left[\frac{-x^2}{2(1-\rho^2)} + \frac{(\rho x)^2}{2(1-\rho^2)}\right] dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{\rho^2 - 1}{2(1-\rho^2)} x^2\right] dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1 \quad \square.$$

Suppose that X and Y are random variables with the standard bivariate normal (or Gaussian) distribution density function f

a) Marginals

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The marginal density function of X

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(y-\rho x)^2 + x^2(1-\rho^2)\right]\right) dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2(1-\rho^2)}(y-\rho x)^2 dy\right]$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}x^2} \sqrt{2\pi(1-\rho^2)}$$

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}$$

$\Rightarrow$ X has the normal distribution with mean $\mathbb{E}(X) = 0$ and variance $var(X) = 1$ (same for Y)

b) Conditional density function:

The condition density function of Y given that $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)$$

$\Rightarrow$ The conditional distribution of Y given $X = x$ is the normal distribution with mean ρx variance $1-\rho^2$

c) Conditional expectation.

The conditional expectation of Y given $X = x$ is

$$\mathbb{E}(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy = \rho x \text{ (follow from b))}$$

d/ Independence

The random variable X and Y are independent
$\Leftrightarrow \quad f(x,y) = g(x)\,h(y) \quad \Leftrightarrow \quad \rho = 0$

Consider the expectation of XY

$$\mathbb{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{Y|X}(y|x)\,f_X(x)\,dx\,dy = \int_{-\infty}^{\infty} x\underbrace{\left[\int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy\right]}_{\mathbb{E}(Y|X=x)} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x\,\underbrace{\mathbb{E}(Y|X=x)}_{\rho x}\, f_X(x)\,dx$$

$$= \rho \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\,dx}_{=}$$

$$= \rho\ \mathbb{E}(X^2)$$

$$= \rho\ \mathrm{var}(X) \quad (\text{since } \mathbb{E}(X) = 0)$$

$$= \rho \quad (\text{since } \mathrm{var}(X) = 1)$$

Also, $\mathbb{E}(X) = \mathbb{E}(Y) = 0$

$\Rightarrow \quad \mathbb{E}(XY) - \mathbb{E}(X) \times \mathbb{E}(Y) = \rho.$

X, Y are independent $\Leftrightarrow \rho = 0$

$\Leftrightarrow \quad \mathbb{E}(X,Y) = \mathbb{E}(X)\,\mathbb{E}(Y)$

* A more general bivariate distribution

Let g be a function of 2 variables :

$$g(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{\frac{-1}{2}Q(x,y)} \quad \text{for } x,y \in \mathbb{R}. \quad (*)$$

where Q is quadratic :

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

$\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$, $-1 < \rho < 1$.

By setting $\begin{cases} \mu_1 = \mu_2 = 0 \\ \sigma_1 = \sigma_2 = 1 \end{cases}$, we obtain the standard bivarate normal distribution

Comment 1: If random variables X and Y have joint density function g, then the pair U, V ~~is~~ given by $U = \frac{X-\mu_1}{\sigma_1}$, $V = \frac{Y-\mu_2}{\sigma_2}$ has the standard bivariate normal distribution with parameter ρ

Proof

Set $u = \frac{x-\mu_1}{\sigma_1}$; $v = \frac{y-\mu_2}{\sigma_2}$

$$\Rightarrow Q(u,v) = \frac{1}{1-\rho^2}\left[u^2 - 2\rho uv + v^2\right]$$

$$f(u,v) = g(x,y)\,|J(u,v)| = g \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = g\sigma_1\sigma_2$$

$\Rightarrow g\sigma_1\sigma_2$ is the joint density func of U, V

$$\sigma_1\sigma_2\, g(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(u^2 - 2\rho uv + v^2\right)\right]$$

$\Rightarrow \sigma_1\sigma_2 g$ is the joint density function of 2 random variables U and V that follow the standard bivariate normal distribution with parameter ρ □

Comment 2. Let X, Y have the general bivarate normal distribution ⊛ $\forall a, b \in \mathbb{R}$. Then, the linear combination $aX + bY$ has a univariate normal distribution

Set $Z = aX + bY$

$$\mathbb{E}(Z) = \mathbb{E}(aX + bY) = a\underbrace{\mathbb{E}(X)}_{\mu_1} + b\underbrace{\mathbb{E}(Y)}_{\mu_2} = a\mu_1 + b\mu_2.$$

$$\begin{aligned} Var(Z) &= Var(aX+bY) = \mathbb{E}[(aX+bY)^2] - [\mathbb{E}((aX+bY))]^2 \\ &= a^2\mathbb{E}(X^2) + b^2\mathbb{E}(Y^2) + 2ab\,\mathbb{E}(XY) - a^2\mathbb{E}(X)^2 - b^2\mathbb{E}(Y)^2 \\ &\quad - 2ab\,\mathbb{E}(X)\,\mathbb{E}(Y) \end{aligned}$$

$$= a^2 \, var(X) + b^2 \, var(Y) + 2ab \, cov[X, Y].$$
$$= a^2 \sigma_1^2 + b^2 \sigma_2^2 + 2ab \, \sigma_1 \sigma_2 \rho$$

By an analogy to a univariate normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu/\sigma)^2} \quad \text{with mean } \mathbb{E}(X) = \mu$$
$$var(X) = \sigma^2.$$

we can construct a univariate normal distribution for random variable Z with mean $\mathbb{E}(Z)$ and variance $Var(Z)$ as shown above

$$f(z) = \frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_1\sigma_2\rho}} e^{-\frac{1}{2}\left(\frac{z - (a\mu_1 + b\mu_2)}{\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_1\sigma_2\rho}}\right)^2}$$

Comment 3

It follows from comment 2 that.

More generally, a vector $\vec{X} = (X_1, X_2 \dots X_n)$ is said to have a multivariate normal distribution if for all $\vec{a} \in \mathbb{R}^n$, the scalar product $\vec{a} \cdot \vec{X}$ has a univariate normal distribution.