Mean and Variance of Binomial distribution.

Let X on $(\Omega, \mathcal{F}, \mathbb{P})$ be a random variable that follows Binomial distribution $\pi_k = \binom{n}{k} p^k q^{n-k}$ $\quad k = 0, 1, 2, \ldots, n$

$p \in [0,1]$.

$q = 1-p$

1st approach:

- Mean:

$$\mathbb{E}(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k q^{n-k} \quad (\text{Here } n \geq 1 \text{ is assumed})$$

$$\mathbb{E}(X) = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} \quad \text{Let } k' = k-1.$$

$$= np \sum_{k'=0}^{n-1} \frac{(n-1)!}{k'!\,(n-1-k')!} p^{k'} q^{n-1-k'}$$

$$= np\,(p+q)^{n-1}$$

$$= np.$$

- If $n = 0$, then $k = 0$. It follows that $\mathbb{E}(X) = np = 0$.

$\therefore \boxed{\mathbb{E}(X) = np} \quad (\forall n)$

- Variance

$$\mathbb{E}(X^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} knp \binom{n-1}{k-1} p^{k-1} q^{n-k} \quad (n \geq 1 \text{ is assumed})$$

Let $k' = k-1$.

$$\mathbb{E}(X^2) = np \sum_{k'=0}^{n-1} (k'+1) \binom{n-1}{k'} p^{k'} q^{n-k'-1}$$

$$= np \left[\sum_{k'=0}^{n-1} \binom{n-1}{k'} p^{k'} q^{n-1-k'} + \sum_{k'=0}^{n-1} k' \binom{n-1}{k'} p^{k'} q^{n-1-k'} \right]$$

$$= np \left[(p+q)^{n-1} + (n-1)p \right] \quad (n \geq 2 \text{ is assumed})$$

$$= np\,[1 + np - p]$$

$$= np(np+q)$$

$$= n^2p^2 + npq$$

- If $n = 0$, then $k = 0 \Rightarrow \mathbb{E}(X^2) = n^2p^2 + npq \, (=0)$.

- If $n = 1$, then $k = 0, 1$

$$\Rightarrow \mathbb{E}(X^2) = 0 + 1^2 \binom{1}{1} p^1 q^0 = p =$$

$$n^2p^2 + npq = p^2 + pq = p(p+q) = p$$

$$\Rightarrow \mathbb{E}(X^2) = n^2p^2 + npq \quad (=p)$$

$$\therefore \quad \mathbb{E}(X^2) = n^2p^2 + npq \quad (\forall n).$$

$$\Rightarrow Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = npq.$$

$$\therefore \quad \boxed{Var(X) = npq}$$

* 2^{nd} approach: Trick of differentiation

General derivation of "m^{th} moment of X":

$$\mathbb{E}(X^m) := \sum_{k=0}^{n} k^m \binom{n}{k} p^k q^{n-k}$$

Let m be some positive integers and let $f_m(x) := \sum_{k=0}^{n} k^m \binom{n}{k} x^k q^{n-k}$

Observe that:

$$\left(x\frac{d}{dx}\right)^m x^k = k^m x^k \quad \forall k.$$

$$\Rightarrow f_m(x) = \sum_{k=0}^{n} \left(x\frac{d}{dx}\right)^m x^k \binom{n}{k} q^{n-k}$$

$$= \left(x\frac{d}{dx}\right)^m \sum_{k=0}^{n} \binom{n}{k} x^k q^{n-k}.$$

$$= \left(x\frac{d}{dx}\right)^m (x+q)^n.$$

In particular, if $n \geq 2$

$$f_1(x) = x\frac{d}{dx}(x+q)^n = nx(x+q)^{n-1}$$

$$f_2(x) = \left(x\frac{d}{dx}\right)^2 (x+q)^n = x\frac{d}{dx}\left(nx(x+q)^{n-1}\right)$$

$$= x\left(n(x+q)^{n-1} + n(n-1)x(x+q)^{n-2}\right).$$

$$= nx(x+q)^{n-1} + n(n-1)x^2(x+q)^{n-2}$$

$$\Rightarrow \mathbb{E}(X) = f_1(p) = np(p+q)^{n-1} = np$$

$$\mathbb{E}(X^2) = f_2(p) = np(p+q)^{n-1} + n(n-1)p^2(p+q)^{n-2}$$

$$= np + n^2p^2 - np^2$$

$$= n^2p^2 + npq$$

& These also holds true for $n = 0, 1$ as previously shown.