Exercise about Joint density functions

Yesui Bataar, PENG Qi

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Question

Random variables X and Y have joint density function

$$f(x,y) = \begin{cases} e^{-x-y} & x,y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{P}(X+Y \leq 1)$ and $\mathbb{P}(X > Y)$

Solution

Derive the probability $\mathbb{P}(X+Y \leq 1)$ by calculating the integral:

$$\begin{aligned} \mathbb{P}(X+Y \leq 1) = & \int_0^1 [\int_0^{1-x} e^{-x-y} dy]\, dx \\ = & \int_0^1 e^{-x}(-e^{-y}|_0^{1-x})\, dx \\ = & \int_0^1 e^{-x}(1-e^{x-1}) dx \\ = & \int_0^1 (e^{-x} - e^{-1})\, dx \\ = & -e^{-x} - e^{-1}x|_0^1 \\ = & (1-e^{-1}) - e^{-1} \\ = & 1 - 2e^{-1}. \end{aligned}$$

Therefore, we derived that $\mathbb{P}(X+Y \leq 1) = 1 - 2e^{-1}$.

Now, find $\mathbb{P}(X > Y)$:
For $X > Y$, we derived the boundary condition in this case that y goes from 0 to ∞, as x goes from y to ∞, as x be greater than y, for all $y > 0$.

Then calculate the integral

$$
\begin{aligned}
\mathbb{P}(X > Y) &= \int_0^\infty \int_y^\infty e^{-x-y} dx\, dy \\
&= \int_0^\infty [\int_y^\infty e^{-x} dx]\, e^{-y} dy \\
&= \int_0^\infty e^{-y} (-e^{-x}|_y^\infty) dy \\
&= \int_0^\infty e^{-y} e^{-y} dy = \int_0^\infty e^{-2y} dy \\
&= \frac{e^{-2y}}{-2}|_0^\infty \\
&= 0 + \frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

Therefore, we derived that $\mathbb{P}(X > Y) = \frac{1}{2}$. $\square$