5.4 Some common density functions. Yesui Baatar

$\Gamma(\omega)$ is the gamma function, defined by.

$$\Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{-x}\, dx$$ where ω is positive integers.

Exercise 5.46.

- Show that the gamma function $\Gamma(\omega)$ satisfies $\Gamma(\omega) = (\omega-1)\Gamma(\omega-1)$ for $\omega > 1$.

$$\Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{-x}\, dx$$

Now integrate by parts $\Gamma(\omega) = -x^{\omega-1} e^{-x} \Big|_0^\infty + \int_0^\infty (\omega-1) x^{\omega-2} e^{-x} dx$

$$= (0-0) + (\omega-1)\underbrace{\int_0^\infty x^{\omega-2} e^{-x}\, dx}_{\Gamma(\omega-1)} = \underline{(\omega-1)\Gamma(\omega-1)}$$

- deduce that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3$

set $\omega = n$.

$\Gamma(n) = (n-1)\Gamma(n-1)$
$\Gamma(n-1) = (n-2)\Gamma(n-2)$
$\vdots$
$\Gamma(2) = 1\,\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = e^{-x}\Big|_\infty^0 = 1.$

$\Gamma(n) = (n-1)(n-2).(n-3). \;.\; \Gamma(1) = (n-1)!$

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Exercise 5.47.

Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx.$

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$$

In polar coordinates : $x = r\cos\vartheta$
$y = r\sin\vartheta.$
$dxdy = rdrd\vartheta.$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2\cos^2\vartheta - r^2\sin^2\vartheta} rdrd\vartheta = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\vartheta =$$

$$\int_0^{2\pi} d\vartheta \int_0^{\infty} e^{-r^2} \frac{dr^2}{2} = \frac{2\pi}{2}\left(-\frac{e^{-r^2}}{2}\right)\Bigg|_0^{\infty} = \frac{2\pi}{2}\left(1-0\right) = \pi$$

$I^2 = \pi \Rightarrow \quad I = \sqrt{\pi}.$