Exercise on Converges in Mean Square

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Question

Let Z_n be a discrete random variable with mass function

$$\mathbb{P}(Z_n = n^\alpha) = \frac{1}{n}, \quad \mathbb{P}(Z_n = 0) = 1 - \frac{1}{n}.$$

Show Z_n converges to 0 in mean square if and only if $\alpha < \frac{1}{2}$. ($\alpha \in \mathbb{R}^+$)

Solution

According to Definition 8.3, one can get $Z_n \to Z$ in mean square as $n \to \infty$ if

$$\mathbb{E}([Z_n - Z]^2) \to 0,$$

for Z as the (limit) random variable.

In the case of this question, it is given that $Z_n = n^\alpha$ and $Z_n = 0$ are corresponded with probabilities $\mathbb{P}(Z_n = n^\alpha) = \frac{1}{n}$ and $\mathbb{P}(Z_n = 0) = 1 - \frac{1}{n}$.
Thus, for Z=0, we can do the calculation as shown below:

$$\begin{aligned}\mathbb{E}([Z_n - 0]^2) &= (n^\alpha - 0)^2 \cdot \mathbb{P}(Z_n = n^\alpha) + (0-0)^2 \cdot \mathbb{P}(Z_n = 0) \\ &= (n^\alpha - 0)^2 \cdot \frac{1}{n} + (0-0)^2 (1 - \frac{1}{n}) \\ &= \frac{n^{2\alpha}}{n}\end{aligned}$$

Because the derived result converges to 0, it should fulfill the in-equation below:

$$\begin{aligned}\frac{n^{2\alpha}}{n} &< 1, \quad (\alpha \in \mathbb{R}^+) \\ \Rightarrow n^{2\alpha} &< n \\ \Rightarrow 2\alpha &< 1 \\ \Rightarrow \alpha &< \frac{1}{2}.\end{aligned}$$

Thus, we can prove that Z_n converges to 0 in mean square if and only $\alpha < \frac{1}{2}$. ($\alpha \in \mathbb{R}^+$) $\square$