Proof of Lemma 1.17

AUSTEN Vic / 062001895

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Statement (i): For $X, Y \in \mathbb{R}^n$, $|X \cdot Y| \leq \|X\| \, \|Y\|$.

By the definition of the norm and the scalar product, one observes that:

$$|X \cdot Y| \leq \|X\| \, \|Y\| \iff \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}$$

As both sides of the equation are positive numbers, one can square both sides to get:

$$\begin{aligned}
&\left(\left|\sum_{i=1}^{n} x_i y_i\right|\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right) \\
\iff\ & 2\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq 2 \cdot \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right) \\
\iff\ & 2 \cdot \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right)^2 \geq 0 \\
\iff\ & \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{j=1}^{n} y_j^2\right) + \left(\sum_{i=1}^{n} y_i^2\right) \cdot \left(\sum_{j=1}^{n} x_j^2\right) - \left(\sum_{i=1}^{n} 2x_i y_i\right) \cdot \left(\sum_{j=1}^{n} x_j y_j\right) \geq 0
\end{aligned}$$

As $\sum_{j=1}^{n} x_j$,$\sum_{j=1}^{n} y_j$, and $\sum_{j=1}^{n} x_j y_j$ are terms independent of the index i, one has:

$$\begin{aligned}
\iff\ & \left(\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2\right) + \left(\sum_{i=1}^{n} y_i^2 \sum_{j=1}^{n} x_j^2\right) - \left(\sum_{i=1}^{n} 2x_i y_i \sum_{j=1}^{n} x_j y_j\right) \geq 0 \\
\iff\ & \left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2\right) + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} y_i^2 x_j^2\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{n} 2x_i y_i x_j y_j\right) \geq 0 \\
\iff\ & \left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2 - 2x_i y_i x_j y_j + x_j^2 y_i^2\right) \geq 0 \\
\iff\ & \left(\sum_{i=1}^{n}\sum_{j=1}^{n} (x_i y_j - x_j y_i)^2\right) \geq 0
\end{aligned}$$

$x_1, x_2, ..., x_n \in \mathbb{R}$ and $y_1, y_2, ..., y_n \in \mathbb{R}$ implies that $x_i y_j - x_j y_i \in \mathbb{R}$ for all $1 \leq i, j \leq n$, and the fact that the sum of the square(s) of real numbers are always greater or equal than 0, one concludes that the statement above is always true, which, in turn proves that the equivalent original statement is true. **Q.E.D.**

Statement (ii): For $X, Y \in \mathbb{R}^n$, $\|X+Y\| \leq \|X\| + \|Y\|$.

By the definition of the norm and the scalar product, one observes that:

$$\|X+Y\| \leq \|X\| + \|Y\| \iff \sqrt{\sum_{i=1}^{n}(x_i+y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}$$

As both sides of the equation are positive numbers, one can square both sides to get:

$$\left(\sum_{i=1}^{n}(x_i+y_i)^2\right) \leq \left(\sum_{i=1}^{n} x_i^2\right) + \left(\sum_{i=1}^{n} y_i^2\right) + 2 \cdot \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}$$

$$\iff \left(\sum_{i=1}^{n} x_i^2\right) + 2\left(\sum_{i=1}^{n} x_i y_i\right) + \left(\sum_{i=1}^{n} y_i^2\right) \leq \left(\sum_{i=1}^{n} x_i^2\right) + \left(\sum_{i=1}^{n} y_i^2\right) + 2\,\|X\|\,\|Y\|$$

By the definition of the scalar product, one has:

$$\iff 2\,(X \cdot Y) \leq 2\,\|X\|\,\|Y\|$$
$$\iff X \cdot Y \leq \|X\|\,\|Y\|$$

As $X \cdot Y \in \mathbb{R}$, one has that $|X \cdot Y| \geq X \cdot Y$, and by **Statement (i)** (i.e. $|X \cdot Y| \leq \|X\|\,\|Y\|$), one concludes that the statement $X \cdot Y \leq \|X\|\,\|Y\|$ is always true. Therefore, the (equivalent) **Statement (ii)** is also true. **Q.E.D.**

Visualizations of **Statement (iii)** for one acute angle and one obtuse angle

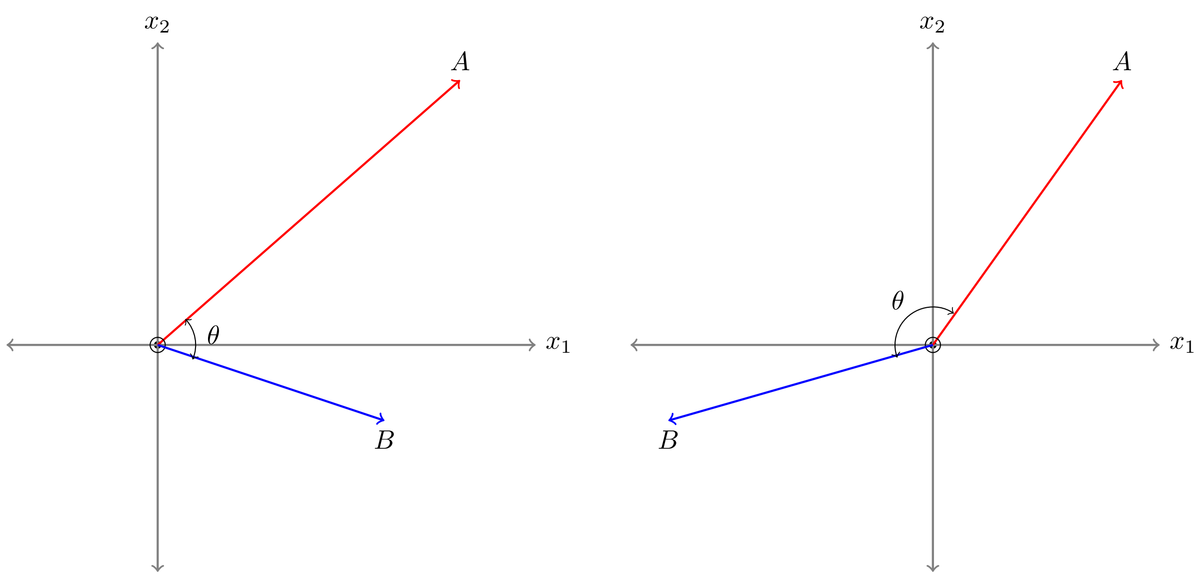

Figure 1: The angle between two vectors A and B in $\mathbb{R}^2$ space, acute (left) and obtuse (right)