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Report 1:

Let $G$ be a weighted and directed network.

Let $S_i, I_j$ the variables of the exact SIS-system.

Let $X_i, Y_j$ —||— approximation system.

If $\forall i = 1, 2, \dots, N$: $\langle I_i \rangle (0) = \langle Y_i \rangle (0)$

Then $\forall t \geq 0$: $\langle I_i \rangle (t) \leq \langle Y_i \rangle (t)$ $(\forall i = 1, \dots, N)$

Proof: An important part of the proof is, that we use

$\forall t \geq 0$: $\langle S_i I_j \rangle (t) \leq \langle S_i \rangle (t) \cdot \langle I_j \rangle (t)$ ①

The book KMS gives [60] as source that this holds everytime for the SIS-system. I found a comment about the book [60], that said, that ① is not ~~proofen~~ proved right in [60]. But I found a book with a proof of ① if we have a small assumption.

<u>Theo:</u> If $\langle S_i I_j \rangle (0) \leq \langle S_i \rangle (0) \cdot \langle I_j \rangle (0)$ holds then $\forall t > 0$: ① hold

The proof is 4 pages long and difficult. I give the idea of this proof.

First check $\langle S_i \rangle \langle I_j \rangle - \langle S_i I_j \rangle = \langle I_i I_j \rangle \langle S_i S_j \rangle - \langle S_i I_j \rangle \langle I_i S_j \rangle$ (easy part)

Second prove it for the graph $\bullet\!\!-\!\!\bullet$ (2 nodes, 1 edge)

Third $A_{ij}(t) := \langle S_i \rangle (t) \cdot \langle I_j \rangle (t) - \langle S_i I_j \rangle (t)$ holds a system of inhomog. lin. differential equations (long + difficult part)

Sort bring this things together. $\square$

So now back to our main proof: We know for the exact

$$\langle \dot{I} \rangle (t) = \tau \sum_{j=1}^{N} g_{ij} \langle S_i I_j \rangle - \gamma \langle I_i \rangle$$

$$\overset{①}{\leq} \tau \sum_{j=1}^{N} g_{ij} \underbrace{\langle S_i \rangle}_{=} \cdot \langle I_j \rangle - \gamma \langle I_i \rangle \qquad (\tau, g_{ij} \geq 0)$$

$$= \tau \sum_{j=1}^{N} g_{ij} \overbrace{(1 - \langle I_i \rangle)} \langle I_j \rangle - \gamma \langle I_i \rangle$$

For the approximation we have

$$\langle \dot{Y}_i \rangle = \tau \sum_{j=1}^{N} g_{ij} (1 - \langle Y_i \rangle) \langle Y_j \rangle - \gamma \langle Y_i \rangle$$

We now use a result of ODE systems, which is called a comparison theorem.

<u>Theo:</u> Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a diff. function

Let $y(t) = f(y(t))$ and $\dot{x}(t) \overset{(*)}{\leq} f(x(t))$ with $x(0) \overset{(*)}{\leq} y(0)$. Let $f$ be cooperative ($:\Leftrightarrow \forall i \neq j \in \{1,..,n\}: \partial_j f_i \geq 0$) in a convex $D \subseteq \mathbb{R}^n$ $D \neq \emptyset$.

ODE

(*) see at page 3 for def. of $\leq$ for vectors

Then $\forall t \geq 0: x(t) \overset{(*)}{\leq} y(t)$

Remark: $f$ be cooperative in a convex $D \subseteq \mathbb{R}^n$, $D \neq \emptyset$

$\Leftrightarrow$ $f$ satisfies Kamke Müller condition

($:\Leftrightarrow \forall i = 1,..,n \ \forall x, y \in D, x \leq y: x_i = y_i \Rightarrow f_i(x) \leq f_i(y)$)

We now show the assumptions from this Theo. to our prob.

$$f: [0,1]^N \subseteq \mathbb{R}^N \to \mathbb{R}^N$$

$$x \mapsto f_i(x) = \tau \sum_{j=1}^{N} g_{ij} (1 - x_i) x_j - \gamma x_i$$

$$\langle \dot{I} \rangle := \begin{pmatrix} \langle \dot{I}_1 \rangle \\ \vdots \\ \langle \dot{I}_N \rangle \end{pmatrix} \leq \begin{pmatrix} \tau \sum g_{ij} (1 - \langle I_i \rangle) \langle I_j \rangle - \gamma \langle I_i \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1(\langle I \rangle) \\ \vdots \\ f_N(\langle I \rangle) \end{pmatrix}$$

$$= f(\langle I \rangle)$$

$\Rightarrow \langle \dot{Y} \rangle = f(\langle Y \rangle)$ and $\langle \dot{I} \rangle \leq f(\langle I \rangle)$

and $\forall i = 1,..,N: \langle I_i \rangle (0) = \langle Y_i \rangle (0)$

$$\forall i \neq k \in \{1,..,N\}: \partial_k f_i(x) = \partial_k \left[ \tau \sum_j g_{ij} (1 - x_i) x_j - \gamma x_i \right]$$

$$= \tau g_{ik} (1 - x_i) \geq 0 \quad (\tau, g_{ik} \geq 0 \text{ and } x_i \in [0,1])$$

$g_{ik}$ is a fixed constant

$\langle I \rangle$ are prob., so $\in [0,1]$ and therefore we defined $f$ only on $[0,1]^N$ and ~~because~~ $[0,1]^N \neq \emptyset$ is convex

$\Rightarrow f$ is cooperative on $[0,1]^N$

$\Rightarrow$ We can use the Theo. and get ( $y := \langle Y \rangle$ ; $x := \langle I \rangle$ )

$\forall t \geq 0: \quad \langle I \rangle(t) \leq \langle Y \rangle(t) \quad \Leftrightarrow$

$\forall t \geq 0: \forall i = 1, \dots, N: \quad \langle I_i \rangle(t) \leq \langle Y_i \rangle(t)$

<u>Remarks:</u> • The proof works for $\gamma_i$ and $\tau_i$ as well.
So we can choose for every node a different $\gamma_i$ or $\tau_i$.

• In general we can get such an upper bound for more approximations systems. With a function $W(x,y) \in \mathbb{R}$ and some assumptions we can get an upper bound and lower bound for the exact system.
Here in this case we have $W(x,y) = x \cdot y$, so the assumption is $\langle S_i I_j \rangle(t) \leq W(\langle S_i \rangle, \langle I_j \rangle)$.
In the paper, where I found this, they say, that $W(x,y) = x \cdot y$ is the most important closure relation.
To have only an upper bound, can be very bad, so it would be very good to have an lower bound, to know how big the error part is.
With $W(x,y) = \min(x,y)$ the paper can find an lower bound.
For special graphs this $W(x,y)$ can be chosen different to get better results.

(*) We use the following ordering relations for vectors

$u, w \in \mathbb{R}^n \qquad u \leq w \;:\Leftrightarrow\; \forall i = 1,2,\dots,n: \quad u_i \underset{\uparrow}{\leq} w_i$

the regular inequality for real numbers

and $\qquad u < w \;:\Leftrightarrow\; \forall i = 1,2,\dots,n: \quad u_i < w_i$