Homework 8

Exercise 1 *Consider the map*

$$f : \mathbb{R}^2 \ni \ (x, y) \mapsto x^3 - 2xy + 2y^2 - 1 \ \in \mathbb{R}.$$

(i) Show that the implicit function theorem can be applied at the point $(1, 1) \in \mathbb{R}^2$*,*

(ii) Compute the tangent at the point $(1, 1)$ *of the curve of equation* $f(x, y) = 0$*, and determine the position of this curve with respect to the tangent line at this point.*

Exercise 2 *Consider the map*

$$f : \mathbb{R}^2 \ni \ (x, y) \mapsto \arctan(x + y) + \mathrm{e}^x - 2y - 1 \ \in \mathbb{R} \ .$$

(i) Show that the implicit function theorem can be applied at any point $(x, y) \in \mathbb{R}^2$ *which satisfies* $f(x, y) = 0$*,*

(ii) Let ϕ *be the function which expresses the second coordinates in terms of the first coordinate, and whose existence is justified by the point (i). Compute the Taylor expansion of* ϕ *up to the order* 2 *near* $(x, y) = (0, 0)$*.*

Exercise 3 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be of class* C^1 *and let* $(x_0, y_0) \in \mathbb{R}^2$ *be a solution of* $f(x_0, y_0) = 0$*. Suppose that* $\partial_y f(x_0, y_0) \neq 0$*. Let* $\phi : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ *be the implicit function of class* C^1 *satisfying* $f\big(x, \phi(x)\big) = 0$ *for any* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ *and satisfying* $\phi(x_0) = y_0$*. Show that*

$$\phi'(x) = -\frac{\big[\partial_x f\big]\big(x, \phi(x)\big)}{\big[\partial_y f\big]\big(x, \phi(x)\big)}$$

whenever the denominator is not 0