Nagoya University, G30 program Spring 2021

Mathematics Tutorial IIa ϟ ☹ ϟ Instructor : Serge Richard

Homework 7

Exercise 1 *a) Let* $\Omega \subset \mathbb{R}^2$ *be open and let* $F : \Omega \to \mathbb{R}^2$ *be of class* C^1 *on* Ω*. Let us also set* $F = (f_1, f_2)$*. Show that if*

$$\partial_x f_2 \neq \partial_y f_1$$

then F *does not admit a potential function of class* C^2*.*

b) What would be a similar statement for a function $F : \Omega \to \mathbb{R}^3$ *if* Ω *is an open subset of* $\mathbb{R}^3$*.*

c) What about the n*-dimensional case, and how many conditions have to be satisfied ?*

Exercise 2 (Spherical coordinates) *Consider the map* $\Phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ *with*

$$\Phi(r, \theta, \varphi) := \big(r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)\big).$$

Compute the Jacobian matrix corresponding to this function.

Exercise 3 *Let us consider* $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $f(x, y, z) = \mathrm{e}^{xy}\cos(z)$ *for any* $(x, y, z) \in \mathbb{R}^3$*. Assume also that* $x = tu$*,* $y = \sin(tu)$ *and* $z = u^2$ *for some* $t, u \in \mathbb{R}$*. By setting*

$$F(t, u) := f\big(tu, \sin(tu), u^2\big)$$

Compute the derivative $\partial_2 F \equiv \partial_u F$ *by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature*

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

Can you explain where this formula comes from ? Can you also understand the (horrible) formula

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$