

## Homework 2

**Exercise 1** Consider the spiral in  $\mathbb{R}^3$  defined by the function

$$f : \mathbb{R} \ni t \mapsto (\cos(t), \sin(t), t) \in \mathbb{R}^3.$$

Find the length of the spiral between  $t = 0$  and  $t = 1$ .

**Exercise 2** Let  $f : (a, b) \rightarrow \mathbb{R}^d$  be a parametric curve of class  $C^1$ , and let  $\varphi : (c, d) \rightarrow (a, b)$  be of class  $C^1$  and strictly increasing, with  $\varphi(c) = a$  and  $\varphi(d) = b$ . Show that the equality  $L_f = L_{f \circ \varphi}$  holds, or in other words show that the length of the curve does not depend on the parametrization.

**Exercise 3** Let  $\Omega \subset \mathbb{R}^2$  and consider the functions  $f_i : \Omega \rightarrow \mathbb{R}$  defined for  $(x, y) \in \Omega$  by

$$a) f_1(x, y) = xy, \quad b) f_2(x, y) = (x + 1)(y + 3) \quad c) f_3(x, y) = \frac{xy}{x^2 + y^2} \quad d) f_4(x, y) = \frac{x + y}{x - y}.$$

1. Determine the maximal domain  $\Omega$  on which these functions are well defined,
2. Sketch the  $k$ -level sets for these functions.

**Exercise 4** Consider the following functions defined on  $\mathbb{R}^2 \setminus \{(0, 0)\}$

$$a) f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad b) f_2(x, y) = \frac{xy}{x^2 + y^2}, \quad c) f_3(x, y) = \frac{1}{x^2 + y^2 + 1}.$$

For each of them compute the limits  $\lim_{x \rightarrow 0} (\lim_{y \rightarrow 0} f_i(x, y))$ ,  $\lim_{y \rightarrow 0} (\lim_{x \rightarrow 0} f_i(x, y))$ , and  $\lim_{(x, y) \rightarrow (0, 0)} f_i(x, y)$ . Discuss your result.

**Exercise 5** Consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) := \begin{cases} \frac{x^2 y}{x^4 - 2x^2 y + 3y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

1. Study the limit  $(x, y) \rightarrow (0, 0)$  along the path of equation  $y = mx$  for any  $m \in \mathbb{R}$ ,
2. Study the limit  $(x, y) \rightarrow (0, 0)$  along the path of equation  $y = x^2$ ,
3. Show that  $f$  is not continuous at  $(0, 0)$ .