http://www.math.nagoya-u.ac.jp/~richard/teaching/s2018/Integrals.pdf

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Homework 12

Let Ω be a body in $\mathbb{R}^n$ and let $\varrho : \Omega \to \mathbb{R}_+$ denote its density function (for $X \in \Omega$ the value $\varrho(X)$ denotes the density of Ω at X). Let M denote the total mass of the body, and let $\bar{X}$ denote the coordinates of its center of mass. These quantities are defined by

$$M = \int_\Omega \varrho(X)\mathrm{d}X,$$
$$\bar{X} = \frac{1}{M}\int_\Omega X\varrho(X)\mathrm{d}X.$$

Note that the second line represents in fact n equalities.

Exercise 1 *(i) Find the center of mass of the quarter of a unit disc Ω defined in polar coordinates by*

$$\Omega = \big\{\big(r\cos(\theta), r\sin(\theta)\big) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi/2\big\},$$

(ii) Find the z-coordinate of the center of mass of the upper half of a unit ball centered at $0 \in \mathbb{R}^3$.

Exercise 2 *Use Green's theorem to compute the integral $\int_c f$ with $f(x,y) = (y^2, x)$ when c corresponds to the following curves, taken counterclockwise:*

(i) The square of vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$,

(ii) The square of vertices $(\pm 1, \pm 1)$,

(iii) The circle of radius 1 and centered at $(0,0)$,

(iv) The ellipse of equation $(x/a)^2 + (y/b)^2 = 1$ for some $a, b > 0$.

Exercise 3 *Check the validity of Green's theorem for the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined for $(x,y) \in \mathbb{R}^2$ by $f(x,y) = (2xy, x^2)$ on the domain $\Omega = [-1,2] \times [-1,3] \subset \mathbb{R}^2$.*

Exercise 4 *Consider the parametrization of the sphere of radius $r > 0$ given by $q : [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ with*

$$q(\theta, \varphi) := \begin{pmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix}.$$

Compute the vectors $[\partial_1 q](\theta,\varphi)$, $[\partial_2 q](\theta,\varphi)$, and the vector normal to the sphere at the point $q(\theta,\varphi)$.