Prop. 3.3.2 Let $E$ and $Q$ be unital C*-algs.

(i) If $\varphi, \psi : E \to Q$ are homotopic $\ast$-homoms, then $K_0(\varphi) = K_0(\psi)$

(ii) If $E$ and $Q$ are homotopy equiv., then $K_0(E)$ is isom. to $K_0(Q)$. More specifically, $E \xrightarrow{\varphi} Q \xrightarrow{\psi} C$ is a homotopy between $E$ and $Q$, then $K_0(\varphi) : K_0(E) \to K_0(Q)$ and $K_0(\psi) : K_0(Q) \to K_0(E)$ are isom. with $K_0(\varphi)^{-1} = K_0(\psi)$

proof

Let $F$ be a path of $\ast$-homoms $t \mapsto F(t)$ with $F(0) = \varphi$, $F(1) = \psi$ s.t. $\forall a \in E$, $[0,1] \ni t \mapsto F(t)a \in Q$ is conti.

For each $n \in \mathbb{N}$, we extend $F$ to a ptwise
Conti. path of */-homom \( F(t) : \text{Mn}(\mathbb{C}) \to \text{Mn}(\mathbb{C}) \)

Then \( \forall p \in \text{Pn}(\mathbb{C}) \), \( t \mapsto F(t)p \) is conti.

We obtain

\[
\psi(p) = F(0)p \sim h F(1)p = \psi(p).
\]

\[
k_\psi(\psi)(p) = [\psi(p)]_0 = [\psi(p)]_0
\]

\[
= k_\psi(p)[p].
\]

Therefore, \( k_\psi(\psi) = k_\psi(p) \)

(ii) Combining (i), Prop.3.3.1 (i) and (ii),

\[
k_\psi(p) \circ k_\psi(\psi)[p]_0
\]

\[
= k_\psi(\psi \circ \psi)[p]_0
\]

\[
= k_\psi(\text{id}_\psi)[p]_0
\]

\[
= \text{id}_k_\psi(\psi)[p]_0 = [p]_0.
\]

Similarly \( k_\psi(p) \circ k_\psi(\psi) = \text{id}_k_\psi(\psi) \)

\( \square \)