About Unitization of $C^*$-algebras

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**Definition 1.** Let $A$ be a $C^*$-algebra with or without a unit. Its (smallest) unitization $\tilde{A}$ is any unital $C^*$-algebra containing $A$ as an ideal and satisfying the following universal property: for any unital $C^*$-algebra $B$ and any *-homomorphism $f : A \to B$, there exists a unique unital *-homomorphism $\tilde{f} : \tilde{A} \to B$ with $\tilde{f} \circ \iota = f$, where $\iota : A \to \tilde{A}$ is the inclusion map.

\[ \begin{array}{ccc} 
\tilde{A} & \xrightarrow{\tilde{f}} & B \\
\iota & & \\
A & \xrightarrow{f} & B
\end{array} \]

**Proposition 2.** For any $C^*$-algebra $A$ with or without a unit, there exists its unitization $\tilde{A}$, and $\tilde{A}$ is unique up to an isomorphism.

**Proof.** First we show the uniqueness. Let $\tilde{A}$ and $\tilde{A}'$ be two unitizations of $A$. By the universal property, there exist $\phi : \tilde{A} \to \tilde{A}'$ and $\psi : \tilde{A}' \to \tilde{A}$ with $\phi \circ \iota_{\tilde{A}} = \iota_{\tilde{A}'}$ and $\psi \circ \iota_{\tilde{A}'} = \iota_{\tilde{A}}$. Then $\psi \circ \phi : \tilde{A} \to \tilde{A}$ and $\phi \circ \psi : \tilde{A}' \to \tilde{A}'$ satisfy $\psi \circ \phi \circ \iota_{\tilde{A}} = \iota_{\tilde{A}'}$ and $\phi \circ \psi \circ \iota_{\tilde{A}'} = \iota_{\tilde{A}}$. On the other hand, $\text{id}_{\tilde{A}} : \tilde{A} \to \tilde{A}$ and $\text{id}_{\tilde{A}'} : \tilde{A}' \to \tilde{A}'$ satisfy $\text{id}_{\tilde{A}} \circ \iota_{\tilde{A}} = \iota_{\tilde{A}}$ and $\text{id}_{\tilde{A}'} \circ \iota_{\tilde{A}'} = \iota_{\tilde{A}}$. By the uniqueness in the universal property, $\phi \circ \psi = \text{id}_{\tilde{A}}$ and $\psi \circ \phi = \text{id}_{\tilde{A}'}$. Then $\phi : \tilde{A} \to \tilde{A}'$ is a *-isomorphism with $\phi \circ \iota_{\tilde{A}} = \iota_{\tilde{A}'}$. Then the unitization is unique up to an isomorphism.

Next we show the existence. The case $A$ is unital is clear. Then we consider the case $A$ is non-unital. Set $\tilde{A} = A \oplus \mathbb{C}$ as a direct sum of complex vector spaces and define the multiplication and involution on $\tilde{A}$ by

\[(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha \beta), \quad (a, \alpha)^* = (a^*, \alpha).\]

Let $\| \cdot \|_A$ be the norm of $A$. For each $x = (a, \alpha) \in \tilde{A}$,

\[
\|x\|_{\tilde{A}} = \sup_{b \in A, \|b\|_A \leq 1} \|ab + \alpha b\|_A, \quad \|x\|_{\tilde{A}} = \max\{\|x\|_{\tilde{A}}, |\alpha|\}.
\]
Then \((\widetilde{A}, \| \cdot \|_{\infty})\) is a unital \(C^*\)-algebra with unit \(1_{\widetilde{A}} = (0, 1)\). We define the \(*\)-homomorphisms

\[\iota: A \to \widetilde{A}, \quad \pi: \widetilde{A} \to \mathbb{C}, \quad \lambda: \mathbb{C} \to \widetilde{A}\]

by

\[\iota(a) = (a, 0), \quad \pi(a, \alpha) = \alpha, \quad \lambda(\alpha) = (0, \alpha)\]

Then the following sequence is a split short exact sequence of \(C^*\)-algebras:

\[0 \to A \xrightarrow{\iota} \widetilde{A} \xrightarrow{\pi} \mathbb{C} \to 0.\]

This construction of \(\widetilde{A}\) is valid even when \(A\) is unital, and \(\widetilde{A}\) is isomorphic to \(A \oplus \mathbb{C}\) as \(C^*\)-algebras if and only if \(A\) is unital. But \(\widetilde{A}\) is not the smallest unitization. We will prove that this construction is well-defined in the last part of this report. By identifying \(\iota\) with the inclusion map and identifying \(\pi\) with the quotient map, \(\widetilde{A}\) is the unitization of \(A\). Indeed, for any unital \(C^*\)-algebra \(B\) with unit \(1_B\) and any \(*\)-homomorphism \(f: A \to B\), we define \(\widetilde{f}: \widetilde{A} \to B\) by \(\widetilde{f}(a, \alpha) = a + \alpha 1_B\). Then \(\widetilde{f}\) is clearly a unital \(*\)-homomorphism with \(\widetilde{f} \circ \iota = f\). If \(\widetilde{f}': \widetilde{A} \to B\) is another unital \(*\)-homomorphism with \(\widetilde{f}' \circ \iota = f\), then for any \((a, \alpha) \in \widetilde{A}\),

\[\widetilde{f}'(a, \alpha) = \widetilde{f}' \circ \iota(a) + \alpha 1_B = f(a) + \alpha 1_B = \widetilde{f} \circ \iota(a) + \alpha 1_B = \widetilde{f}(a, \alpha).\]

Then \(\widetilde{f}' = \widetilde{f}\) and so \(\widetilde{f}\) is unique. \qed

By the universal property, the following holds.

**Corollary 3.** For any \(*\)-homomorphism \(f: A \to B\), there exists a unique unital \(*\)-homomorphism \(\widetilde{f}: \widetilde{A} \to B\) such that \(\widetilde{f} \circ \iota_A = \iota_B \circ f\). In other words, the unitization of \(C^*\)-algebras is functorial.

**Example 4.** Let \(X\) be a non-compact and locally compact Hausdorff space. Then \((C_0(X), \| \cdot \|_{\infty})\) is a non-unital \(C^*\)-algebra. The unitization of \((C_0(X), \| \cdot \|_{\infty})\) is isomorphic to \((C(X^+) + \| \cdot \|_{\infty})\) where \(X^+\) is the one-point compactification of \(X\). In this sense, the unitization of \(C^*\)-algebras is the non-commutative analogue of one-point compactification of non-compact and locally compact Hausdorff spaces.

We considered the smallest unitization of \(C^*\)-algebras so far. But there exists a notion of "the largest" unitization of \(C^*\)-algebras. We introduce it without proof.

**Definition 5.** Let \(A\) be a \(C^*\)-algebra. An ideal \(I\) of \(A\) is essential if \(I \cap J \neq 0\) for any non-zero ideal \(J\) of \(A\).

**Definition 6.** Let \(A\) be \(C^*\)-algebra. Its multiplier algebra \(M(A)\) is any unital \(C^*\)-algebra containing \(A\) as an essential ideal and satisfying the following universal property: for any \(C^*\)-algebra \(B\) containing \(A\) as an essential ideal, there exists a unique \(*\)-homomorphism \(\phi: B \to M(A)\) such that \(\phi \circ \iota_B = \iota_{M(A)}\) and where \(\iota_B: A \to B\) and \(\iota_{M(A)}: A \to M(A)\) are the inclusion maps.
For any $C^*$-algebra $A$ with or without a unit, there exists its multiplier $M(A)$, and $M(A)$ is unique up to an isomorphism.

The multiplier algebra $M(A)$ can be constructed as the set of double centralizer of $A$ (see [5]).

**Example 8.** Let $X$ be a locally compact Hausdorff space. Let $A = (C_0(X), \| \cdot \|_\infty)$ be a $C^*$-algebra. Then $M(A)$ is isomorphic to $(C_0(X), \| \cdot \|_\infty)$. On the other hand, $C_0(X)$ is isomorphic to $C(\beta X)$ where $\beta X$ is the Stone-Čech compactification of $X$. In this sense, the multiplier of $C^*$-algebras is the non-commutative analogue of the Stone-Čech compactification of locally compact Hausdorff spaces.

**Example 9.** Let $\mathcal{H}$ be a separable Hilbert space and let $A = \mathcal{K}(\mathcal{H})$ be the $C^*$-algebra of all compact operators on $\mathcal{H}$. Then $M(A)$ is isomorphic to $\mathcal{B}(\mathcal{H})$, the $C^*$-algebra of all bounded operators on $\mathcal{H}$.

Finally, we prove that the unitization $\tilde{A}$ of $A$ constructed above is a $C^*$-algebra.

**Proof.** First, we show that $\| \cdot \|_{\tilde{A}}$ is a norm. We only show that if $\| x \|_{\tilde{A}} = 0$, then $x = 0$. The rest of the condition for the norm is clear since $\| \cdot \|_{A}, \| \cdot \|$ are norms. If $x = (a, \alpha) \in \tilde{A}$ with $\| x \|_{\tilde{A}} = 0$, then $\| x \|_{A} = 0$ and $|\alpha| = 0$. So, $\alpha = 0$ and $\| ab \|_{A} = 0$ for any $b \in A$ with $\| b \|_{A} \leq 1$. By the $C^*$-condition of $\| \cdot \|_{A}$, if $a \neq 0$, $\| a^* \|_{A} = 1$ and

$$
\| a \|_{A} = \frac{\| a \|_{A}^2}{\| a \|_{A}} = \| a a^* \|_{A} = \left\| \alpha \left( \frac{a^*}{\| a \|_{A}} \right) \right\|_{A} = 0.
$$

Since $\| \cdot \|_{A}$ is a norm, we have $a = 0$. Thus $x = 0$. Therefore $\| \cdot \|_{\tilde{A}}$ is a norm.

Next, we show that $\tilde{A}$ is complete with respect to $\| \cdot \|_{\tilde{A}}$. Let $\{ x_n = (a_n, \alpha_n) \}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\tilde{A}$. Then,

$$
| \alpha_n - \alpha_m | \leq \| x_n - x_m \|_{\tilde{A}} \to 0 \quad (n, m \to \infty),
$$

$$
\| a_n - a_m \|_{A} = \| (a_n - a_m)(a_n - a_m)^* \|_{A}^{1/2}
= \begin{cases} 
\| a_n - a_m \|_{A}^{1/2} \left( \frac{\| a_n - a_m \|_{A}}{\| a_n - a_m \|_{A}} \right) & \text{if } a_n - a_m \neq 0 \\
0 & \text{if } a_n - a_m = 0
\end{cases}
\leq \| a_n - a_m \|_{A}^{1/2} \sup_{\| b \|_{A} \leq 1} \| (a_n - a_m)b \|_{A}^{1/2}
= \| a_n - a_m \|_{A}^{1/2} \| (a_n - a_m, 0) \|_{A}^{1/2},
$$

\[ 3 \]
so, one has

\[ \|a_n - a_m\|_A \leq \|(a_n - a_m, 0)\|_{\tilde{A}} \]
\[ \leq \|(a_n - a_m, 0)\|_{\tilde{A}} \]
\[ \leq \|x_n - x_m\|_{\tilde{A}} + \|(0, \alpha_n - \alpha_m)\|_{\tilde{A}} \]
\[ = \|x_n - x_m\|_{\tilde{A}} + |\alpha_n - \alpha_m| \rightarrow 0 \quad (n, m \rightarrow \infty). \]

Then \( \{a_n\}_{n \in A} \) and \( \{\alpha_n\}_{n \in \mathbb{N}} \) are Cauchy sequences in \( A \) and \( \mathbb{C} \) respectively. Since \( A \) and \( \mathbb{C} \) are complete, there exist \( a \in A \) and \( \alpha \in \mathbb{C} \) such that \( \|a_n - a\|_A \rightarrow 0 \) and \( |\alpha_n - \alpha| \rightarrow 0 \) as \( n \rightarrow \infty \). Since

\[ \|x_n - x\|_{\tilde{A}} = \sup_{\|b\|_A \leq 1} \|(a_n - a)b + (\alpha_n - \alpha)b\|_A \]
\[ \leq \sup_{\|b\|_A \leq 1} (\|a_n - a\|_A\|b\|_A + |\alpha_n - \alpha|\|b\|_A) \]
\[ = \|a_n - a\|_A + |\alpha_n - \alpha|, \]

then, one has

\[ \|x_n - x\|_{\tilde{A}} = \|(a_n - a, \alpha_n - \alpha)\|_{\tilde{A}} \]
\[ = \max\{\|x_n - x\|_{\tilde{A}}, |\alpha_n - \alpha|\} \]
\[ \leq \max\{\|a_n - a\|_A + |\alpha_n - \alpha|, |\alpha_n - \alpha|\} \rightarrow 0 \quad (n \rightarrow \infty). \]

Thus \( \tilde{A} \) is complete with respect to \( \|\cdot\|_{\tilde{A}} \).

Next we show that \( (\tilde{A}, \|\cdot\|_{\tilde{A}}) \) is a Banach algebra. For any \( x = (a, \alpha), y = (b, \beta), z = (c, \gamma) \in \tilde{A} \) and \( \zeta \in \mathbb{C} \),

\[ (\zeta x) y = (\zeta a, \zeta \alpha)(b, \beta) = (\zeta ab + \beta \zeta a + \alpha \zeta b, \alpha \beta), \]
\[ \zeta (xy) = (\zeta a + \beta a, \alpha \beta) = (\zeta ab + \zeta \alpha b, \zeta \alpha \beta), \]
\[ x(\zeta y) = (a, \alpha)(\zeta b, \zeta \beta) = (\zeta ab + \beta \zeta a + \alpha \zeta b, \alpha \beta). \]

Then \( (\zeta a)b = \zeta(ab) = a(\zeta b) \).

\[ x(y + z) = (a, \alpha)(b + c, \beta + \gamma) \]
\[ = (a(b + c) + (\beta + \gamma)a + \alpha(b + c), \alpha(\beta + \gamma)) \]
\[ = ((ab + \beta a + \alpha b) + (ac + \gamma a + \alpha c), \alpha \beta + \alpha \gamma) \]
\[ = xy + xz. \]
\[ (x + y)z = (a + b, \alpha + \beta)(c, \gamma) \]
\[ = ((a + b)c + \gamma(a + b) + (\alpha + \beta)c, (\alpha + \beta)\gamma)) \]
\[ = ((ac + \gamma a + \alpha c) + (bc + \gamma b + \beta c), \alpha \gamma + \beta \gamma) \]
\[ = xz + yz. \]
Then \( x(y + z) = xy + xz \) and \( (x + y)z = xz + yz \). Since

\[
\| xy \| \tilde{A} = \| (a, \alpha)(b, \beta) \| \tilde{A} \\
= \| (ab + \beta a + \alpha b, \alpha \beta) \| \tilde{A} \\
= \sup_{\| c \| \tilde{A} \leq 1} \| (ab + \beta a + \alpha b) c + \alpha \beta c \| \tilde{A} \\
= \sup_{\| c \| \tilde{A} \leq 1} \| a(bc + \beta c) + \alpha (bc + \beta c) \| \tilde{A} \\
\leq \| x \| \tilde{A} \sup_{\| c \| \tilde{A} \leq 1} \| bc + \beta c \| \tilde{A} \\
\leq \| x \| \tilde{A} \| y \| \tilde{A},
\]

then, one has

\[
\| xy \| \tilde{A} = \max \{ \| xy \| \tilde{A}, \| \alpha \beta \| \}
\leq \max \{ \| x \| \tilde{A} \| y \| \tilde{A}, \| \alpha \| \| \beta \| \}
\leq \max \{ \| x \| \tilde{A} \| y \| \tilde{A}, \| x \| \tilde{A} \| \beta \|, \| \alpha \| \| \beta \| \}
\leq \max \{ \| x \| \tilde{A}, \| \alpha \| \} \max \{ \| y \| \tilde{A}, \| \beta \| \}
= \| x \| \tilde{A} \| y \| \tilde{A}.
\]

Thus \( \| xy \| \tilde{A} \leq \| x \| \tilde{A} \| y \| \tilde{A} \). So, \((\tilde{A}, \| \cdot \| \tilde{A})\) is a Banach algebra.

Next, we show that \((\tilde{A}, \| \cdot \| \tilde{A})\) is a \( C^* \)-algebra. For any \( x = (a, \alpha), y = (b, \beta) \in \tilde{A} \) and \( \zeta \in \mathbb{C} \),

\[
(x^*)^* = (a^*, \overline{\alpha})^* = (a, \alpha) = x,
\]

Then \((x^*)^* = x\).

\[
(x + y)^* = (a + b, \alpha + \beta)^* = ((a + b)^*, \overline{\alpha + \beta}) = (a^* + b^*, \overline{\alpha} + \overline{\beta}) = x^* + y^*,
\]

Then \((x + y)^* = x^* + y^*\).

\[
(\zeta x)^* = (\zeta a, \zeta \alpha)^* = ((\zeta a)^*, \overline{\zeta \alpha}) = (\overline{\zeta} a^*, \overline{\zeta} \alpha) = \overline{\zeta} (a^*, \alpha) = \overline{\zeta} x^*,
\]

Then \((\zeta x)^* = \overline{\zeta} x^*\).

\[
(xy)^* = (ab + \beta a + \alpha b, \alpha \beta)^* = ((ab + \beta a + \alpha b)^*, \overline{\alpha \beta}) = (b^* a^* + \overline{\beta} a^* + \alpha \overline{b}^*, \overline{\beta} \alpha) = y^* x^*,
\]

Then \((xy)^* = y^* x^*\). Since the \( C^* \)-condition \( \| x^* x \| \tilde{A} = \| x \| \tilde{A}^2 \) on a Banach *-algebra \(\tilde{A}\) is
equivalent to $\|x^*x\|_{\tilde{A}} \geq \|x^*\|_{\tilde{A}}\|x\|_{\tilde{A}}$ and
\[
\|x^*x\|_{\tilde{A}} = \sup_{\|c\|_{\tilde{A}} \leq 1} \|(a^*a + \overline{a}a + \alpha a^*)c + |\alpha|^2 c\|_{\tilde{A}} \\
= \sup_{c \neq 0, \|c\|_{\tilde{A}} \leq 1} \|c^*\|_{\tilde{A}} \|(a^*a + \overline{a}a + \alpha a^*)c + |\alpha|^2 c\|_{\tilde{A}} \\
\geq \sup_{c \neq 0, \|c\|_{\tilde{A}} \leq 1} \|c^*(a^*a + \overline{a}a + \alpha a^*)c + |\alpha|^2 c\|_{\tilde{A}} \\
= \sup_{\|c\|_{\tilde{A}} \leq 1} \|c^*\|_{\tilde{A}} \|(ac + \alpha c^*)(ac + \alpha c)\|_{\tilde{A}} \\
= \|x^*\|_{\tilde{A}}\|x\|_{\tilde{A}},
\]
then, one has
\[
\|x^*x\|_{\tilde{A}} = \|(a^*a + \overline{a}a + \alpha a^*, \overline{\alpha}a)\|_{\tilde{A}} \\
= \max\{\|x^*x\|_{\tilde{A}}, |\overline{\alpha}|\} \\
\geq \max\{\|x^*\|_{\tilde{A}}\|x\|_{\tilde{A}}, |\overline{\alpha}|\} \\
= \|x^*\|_{\tilde{A}}\|x\|_{\tilde{A}}.
\]
Thus $(\tilde{A}, \| \cdot \|_{\tilde{A}})$ is a $C^*$-algebra with unit $1_{\tilde{A}} = (0, 1)$.

If $\tilde{A}$ is isomorphic to $A \oplus \mathbb{C}$ as $C^*$-algebras, then there exists a $^*$-isomorphism $f: \tilde{A} \to A \oplus \mathbb{C}$. For any $(a, \alpha) \in A \oplus \mathbb{C}$, there exists $x \in \tilde{A}$ such that $f(x) = (a, \alpha)$. So,
\[
f(1_{\tilde{A}})(a, \alpha) = f(1_{\tilde{A}})f(x) = f(1_{\tilde{A}}x) = f(x) = (a, \alpha), \\
(a, \alpha)f(1_{\tilde{A}}) = f(x)f(1_{\tilde{A}}) = f(x1_{\tilde{A}}) = f(x) = (a, \alpha).
\]
Then $f(1_{\tilde{A}})$ is a unit of $\tilde{A}$. Let $f(1_{\tilde{A}}) = (b, \beta)$ then for any $(a, \alpha) \in A \oplus \mathbb{C},$
\[
(a, \alpha) = (a, \alpha)f(1_{\tilde{A}}) = (a, \alpha)(b, \beta) = (ab, \alpha \beta), \\
(a, \alpha) = f(1_{\tilde{A}})(a, \alpha) = (b, \beta)(a, \alpha) = (ba, \beta a).
\]
Thus $a = ab = ba$ and $\alpha = \alpha \beta = \beta \alpha$. So, $b$ is a unit in $A$. This implies $A$ is a unital $C^*$-algebra. Conversely If $A$ is a unital $C^*$-algebra with a unit $1_A$, then $f: A \to A \oplus \mathbb{C}$ defined by $f(a, \alpha) = (a + 1_A \alpha, \alpha)$ is a $^*$-isomorphism. Indeed, $g: A \oplus \mathbb{C} \to A$ defined by $g(a, \alpha) = (a - 1_A \alpha, \alpha)$ is an inverse map of $f$ and
\[
f((a, \alpha)(b, \beta)) = f(ab + \beta a + \alpha b, \alpha \beta) \\
= (ab + \beta a + \alpha b + \alpha \beta 1_A, \alpha \beta) \\
= ((a + 1_A \alpha)(b + \beta 1_A), \alpha \beta) \\
= (a + 1_A \alpha)(b + \beta 1_A, \beta) \\
= f(a, \alpha)f(b, \beta),
\]
and

\[ f((a, \alpha)^*) = f(a^*, \overline{\alpha}) = (a^* + \overline{\alpha}1_A, \overline{\alpha}) = ((a + \alpha 1_A)^*, \overline{\alpha}) = (a + \alpha 1_A, \alpha)^* = f(a, \alpha)^*. \]

References


[5] Paul Skoufranis, An introduction to multiplier algebras,  