Exercise 2.1.6

(i) prove $U_0(c) \triangleleft U(c)$

First, $U_0(c)$ is closed w.r.t. group multiplication.

Indeed $U_1, U_2 \in U_0(c)$

$W_1(t) : [0,1] \to U(c), \quad W_1(0) = U_1, W_1(1) = U_1 U_2 \sim 1$ by $W_1 W_2(t), \quad U_1, U_2 \in U_0(c)$

Next, $U_0(c)$ is closed w.r.t. inverse.

Indeed $U \in U_0(c)$

$W(t) : [0,1] \to U(c), \quad W(0) = U, W(1) = 1$

Now we can think $W(t)$ because $W(t) \in U(c)$ and unitary ops are invertible.

$W(t)$ is a path from $U$ to $1$.

Hence, $U U^* \in U_0(c), \quad U \in U(c), U \in U_0(c)$.

$W(t) : [0,1] \to U(c), \quad W(0) = U, W(1) = 1$

$U U^* \sim 1 \iff U^* U = 1 \quad (U \in U(c) \text{ D})$

(ii) prove $U_0(c)$ is open and closed relative to $U(c)$.

$G = \{ \exp(i h_1), \ldots, \exp(i h_n) \mid n \in \mathbb{N}, h_1, \ldots, h_n : \text{self-adj} \}$

by Lemma 2.1.2 (i) and $U_0(c)$ is group (closed w.r.t. multiplication).

so $G \subset U_0(c)$ and $G$ is also a group (easy to see).

Now we will show $G \subset U(c)$ is open.

Indeed $U \in G$ and take $U \in U(c)$ s.t. $\|u - v\| < 2$

by Lemma 2.1.2 (iii) and $\|1 - u v^*\| = \|u - v\| < 2, 1 \sim h u v^*$ and $-I \in \text{sp}(u v^*)$.

In the proof of Lemma 2.1.2 (ii) $h : \text{self-adj}, U^* = e^{ih}$.
So \( U = \exp(i\theta) U \in G \).

This implies \( G \) is open.

Next we will prove \( G \) is closed.

By decomposition of coset

\[
U(c) = \bigcup_{u \in U(c)} L_i Gu
\]

So

\[
U(c) \setminus G = \bigcup_{u \in U(c)} Gu
\]

\( \text{hom} \)

\( G \cong Gu \) (indeed \( u \) is invertible).

And we've already known \( G \) open.

So \( U(c) \setminus G \) is open, hence \( G \) closed. □

(iii) prove \( U(c) = G \)

\( G \) is a non-empty subset of \( U_0(c) \), \( G \) open closed

\( U_0(c) \) is connected (this is clear by def of \( U_0(c) \)).

\( U_0(c) = G \).
$W^{-1}(t)$: Continuous.

Fix $a \in (0, 1)$.

$$
\| W^{-1}(a+t) - W^{-1}(a) \| = \| W^{-1}(a+t) (W(a) - W(a+t)) W^{-1}(a) \|
\leq \| W^{-1}(a+t) \| \| W(a) - W(a+t) \| \| W^{-1}(a) \|
= \| W(a) - W(a+t) \| \quad (\because \|U\| = 1, \ U: \text{unitary})
$$

Since $W(t)$ is cont, $W^{-1}(t)$ is cont.

$W'(t) = U W(t) U^* : \text{cont}$

$$
\| W'(a+t) - W'(t) \| \leq \| U \| \| W(a+t) - W(t) \| \| U^* \|
= \| W(a+t) - W(t) \|
$$