Exercise 2.2.2. Let $\mathcal{C}$ be a unital $C^*$-algebra and let $p$ be a projection in $\mathcal{C}$. Then, we have $\sigma(p) \subset \{0, 1\}$.

(Proof) Clearly we have $\sigma(0) = \{0, 1\}$ and $\sigma(1) = 1$. Hence we consider $p \neq 0, 1$.

Since $p$ is a projection, we have $p(I - p) = 0$, which implies $\sigma(0, 1) \subset \sigma(p)$. Conversely, for $\lambda \neq 0, 1$, we have

$$\left\{ \frac{1}{\lambda(1-\lambda)} \left( p - \frac{1}{\lambda} I \right) \right\}$$

$$= \frac{1}{\lambda(1-\lambda)} \left( p^2 - \frac{1}{\lambda} p - \frac{1}{1-\lambda} p + I \right)$$

$$= \left\{ \frac{1}{\lambda(1-\lambda)} - \frac{1}{\lambda} - \frac{1}{1-\lambda} \right\} p + I \quad (\because p \text{ is a projection})$$

$$= I$$

and hence similarly

$$\left\{ \frac{1}{\lambda(1-\lambda)} \left( p - \frac{1}{\lambda} I \right) \right\} (p - \lambda I) = I$$

This implies $\sigma(p) = \{0, 1\}$. Thus we conclude

$$\sigma(p) = \begin{cases} \{0\} & \text{if } p = 0 \\ \{1\} & \text{if } p = 1 \\ \{0, 1\} & \text{else} \end{cases}.$$