Exercise 1.1.6

(i) \( C, Q : C^* \text{-algebra}, \phi : C \to Q : \ast \text{-homomorphism} \),

\[(i) \ \phi \text{ isometric } \iff \phi \text{ is injective}
\]

(ii) \( \ker \phi : C^* \text{-subalgebra of } C \),
\[\text{ran } \phi : C^* \text{-subalgebra of } Q\]

Proof

\((i) \Rightarrow \)

Let \( \phi \) be isometric.

Then, if \( \phi(a) = \phi(b) \) for \( a, b \in C \),
\[
\|a - b\| = \|\phi(a) - \phi(b)\| \quad (\because \phi \text{ isometric})
\]
\[
= \|\phi(a) - \phi(b)\| \quad (\because \phi \text{ homomorphism})
\]
\[
= 0 \quad (\because \phi(a) = \phi(b))
\]

Hence, \( a = b \).

\((\Leftarrow)\)

Let \( \phi \) is injective.

Let \( \varphi : = \phi^{-1} : \text{Im } \phi \to C \).

\(\forall a \in C\),
\[
\|\phi(a)\|^2 = \|\phi(\phi^{-1} \phi(a))\| \quad (\because Q : C^* \text{-algebra})
\]
\[
= \|\phi(\phi^{-1} \phi(a))\| \quad (\because \phi \text{ homomorphism})
\]
\[
= \|\phi(a)\| \quad (\because a^*a \text{ is hermitian})
\]
\[
\leq \gamma(a^*a) \quad (\because \sigma(\phi(\phi^{-1} \phi(a))) \subset \sigma(a^*a))
\]
\[ = \|a^*a\| \quad (\because a^*a \text{ is hermitian}) \]
\[ = \|a\|^2 \quad (\because \mathbb{C} \text{ is a *-algebra}) \]

So,
\[ \|\varphi(a)\| \leq \|a\| \]

Similarly,
\[ \|a\| = \|\varphi(\varphi(a))\| \leq \|\varphi(a)\| \]

Hence,
\[ \|\varphi(a)\| = \|a\| \]

(ii) \text{ Ker } \varphi : C^* subalgebra of \mathbb{C}.

The only thing that needs to be proved is that
\text{ Ker } \varphi is closed.

Let \( a \in \text{ Ker } \varphi \) (\( n \in \mathbb{N} \)), \( a \in \mathbb{C} \) be an \( a \to a \).

Then \( \forall n \in \mathbb{N} \), \( \varphi(an) = 0 \).

Since \( \varphi \) is continuous,
\[ \varphi(a) = \lim_{n \to \infty} \varphi(an) = 0 \]

Hence, \( a \in \text{ Ker } \varphi \), i.e., \( \text{ Ker } \varphi \) is closed.

\text{ Ran } \varphi : C^* subalgebra of \mathbb{Q}.

Since \( \text{ Ker } \varphi \) is a closed ideal of \( \mathbb{C} \),
\[ \mathbb{C}/\text{ Ker } \varphi \text{ is a } C^* \text{ algebra.} \]

\[ \|a + \text{ Ker } \varphi\| = \inf_{\beta \in \text{ Ker } \varphi} \|a + \beta\|, \]
\[ (a + \text{ Ker } \varphi)^* = a^* + \text{ Ker } \varphi \quad (a \in \mathbb{C}) \]

By a homomorphism theorem,
\[ \mathbb{C}/\text{ Ker } \varphi \cong \text{ Ran } \varphi \]
Hence, $\text{Ran} \Phi$ is a C*-subalgebra of $Q$. \qed