# $K$-theory for $C^*$-algebras, and beyond

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# Contents

# Chapter 1

# $C^*$-algebras

This chapter is mainly based on the first chapters of the books [Mur90] and [RLL00].

## 1.1 Basics on $C^*$-algebras

**Definition 1.1.1.** A Banach algebra $\mathcal{C}$ *is a complex vector space endowed with an associative multiplication and with a norm* $\|\cdot\|$ *which satisfy for any* $a, b, c \in \mathcal{C}$ *and* $\alpha \in \mathbb{C}$

*(i)* $(\alpha a)b = \alpha(ab) = a(\alpha b)$,

*(ii)* $a(b+c) = ab + ac$ *and* $(a+b)c = ac + bc$,

*(iii)* $\|ab\| \leq \|a\|\,\|b\|$ *(submultiplicativity)*

*(iv)* $\mathcal{C}$ *is complete with the norm* $\|\cdot\|$.

One says that $\mathcal{C}$ is *Abelian* or *commutative* if $ab = ba$ for all $a, b \in \mathcal{C}$. One also says that $\mathcal{C}$ is *unital* if $\mathbf{1} \in \mathcal{C}$, *i.e.* if there exists an element $\mathbf{1} \in \mathcal{C}$ such that $\mathbf{1}a = a = a\mathbf{1}$ for all $a \in \mathcal{C}$. *A subalgebra* $\mathcal{J}$ *of* $\mathcal{C}$ is a vector subspace which is stable for the multiplication. If $\mathcal{J}$ is norm closed, it is a Banach algebra in itself.

**Examples 1.1.2.** *(i)* $\mathbb{C}$ *or* $M_n(\mathbb{C})$ *(the set of* $n \times n$*-matrices over* $\mathbb{C}$*) are unital Banach algebras.* $\mathbb{C}$ *is Abelian, but* $M_n(\mathbb{C})$ *is not Abelian for any* $n \geq 2$.

*(ii) The set* $\mathcal{B}(\mathcal{H})$ *of all bounded operators on a Hilbert space* $\mathcal{H}$ *is a unital Banach algebra.*

*(iv) The set* $\mathcal{K}(\mathcal{H})$ *of all compact operators on a Hilbert space* $\mathcal{H}$ *is a Banach algebra. It is unital if and only if* $\mathcal{H}$ *is finite dimensional.*

*(iv) If* $\Omega$ *is a locally compact topological space,* $C_0(\Omega)$ *and* $C_b(\Omega)$ *are Abelian Banach algebras, where* $C_b(\Omega)$ *denotes the set of all bounded and continuous functions from* $\Omega$ *to* $\mathbb{C}$*, and* $C_0(\Omega)$ *denotes the subset of* $C_b(\Omega)$ *of functions* $f$ *which*

*vanish at infinity,* i.e. *for any* $\varepsilon > 0$ *there exists a compact set* $K \subset \Omega$ *such that* $\sup_{x\in\Omega\setminus K} |f(x)| \leq \varepsilon$. *These algebras are endowed with the* $L^\infty$*-norm, namely* $\|f\| = \sup_{x\in\Omega} |f(x)|$. *Note that* $C_b(\Omega)$ *is unital, while* $C_0(\Omega)$ *is not, except if* $\Omega$ *is compact. In this case, one has* $C_0(\Omega) = C(\Omega) = C_b(\Omega)$.

(v) *If* $(\Omega, \mu)$ *is a measure space, then* $L^\infty(\Omega)$, *the (equivalent classes of) essentially bounded complex functions on* $\Omega$ *is a unital Abelian Banach algebra with the essential supremum norm* $\|\cdot\|_\infty$.

Observe that $\mathbb{C}$ is endowed with the complex conjugation, that $M_n(\mathbb{C})$ is also endowed with an operation consisting of taking the transpose of the matrix, and then the complex conjugate of each entry, and that $C_0(\Omega)$ and $C_b(\Omega)$ are also endowed with the operation consisting in taking the complex conjugate $f \mapsto \overline{f}$. All these additional structures are examples of the following structure:

**Definition 1.1.3.** *A* $C^*$-algebra *is a Banach algebra* $\mathcal{C}$ *together with a map* $^* : \mathcal{C} \to \mathcal{C}$ *which satisfies for any* $a, b \in \mathcal{C}$ *and* $\alpha \in \mathbb{C}$

(i) $(a^*)^* = a$,

(ii) $(a + b)^* = a^* + b^*$,

(iii) $(\alpha a)^* = \overline{\alpha} a^*$,

(iv) $(ab)^* = b^* a^*$,

(v) $\|a^* a\| = \|a\|^2$.

*The map* $^*$ *is called* an involution.

Clearly, if $\mathcal{C}$ is a unital $C^*$-algebra, then $\mathbf{1}^* = \mathbf{1}$.

**Examples 1.1.4.** *The Banach algebras described in Examples 1.1.2 are in fact* $C^*$*-algebras, once complex conjugation is considered as the involution for complex functions. Note that for* $\mathcal{B}(\mathcal{H})$ *and* $\mathcal{K}(\mathcal{H})$ *the involution consists in taking the adjoint*[1] *of any element* $a \in \mathcal{B}(\mathcal{H})$ *or* $a \in \mathcal{K}(\mathcal{H})$. *In addition, let us observe that for a family* $\{\mathcal{C}_i\}_{i\in I}$ *of* $C^*$*-algebras, the direct sum* $\oplus_{i\in I}\mathcal{C}_i$, *with the pointwise multiplication and involution, and the supremum norm, is also a* $C^*$*-algebra.*

**Definition 1.1.5.** A $*$-homomorphism $\varphi$ *between two* $C^*$*-algebras* $\mathcal{C}$ *and* $\mathcal{Q}$ *is a linear map* $\varphi : \mathcal{C} \to \mathcal{Q}$ *which satisfies* $\varphi(ab) = \varphi(a)\varphi(b)$ *and* $\varphi(a^*) = \varphi(a)^*$ *for all* $a, b \in \mathcal{C}$. *If* $\mathcal{C}$ *and* $\mathcal{Q}$ *are unital and if* $\varphi(\mathbf{1}) = \mathbf{1}$, *one says that* $\varphi$ *is* unit preserving *or* a unital $*$-homomorphism. *If* $\|\varphi(a)\| = \|a\|$ *for any* $a \in \mathcal{C}$, *the* $*$*-homomorphism is* isometric.

[1] If $\mathcal{H}$ is a Hilbert space with scalar product denoted by $\langle\cdot,\cdot\rangle$ and if $a \in \mathcal{B}(\mathcal{H})$, then its adjoint $a^*$ is defined by the equality $\langle af, g\rangle = \langle f, a^* g\rangle$ for any $f, g \in \mathcal{H}$. If $a \in \mathcal{K}(\mathcal{H})$, then $a^* \in \mathcal{K}(\mathcal{H})$ as well.

A *$C^*$-subalgebra* of a $C^*$-algebra $\mathcal{C}$ is a norm closed (non-empty) subalgebra of $\mathcal{C}$ which is stable for the involution. It is clearly a $C^*$-algebra in itself. In particular, if $F$ is a subset of a $C^*$-algebra $\mathcal{C}$, we denote by $C^*(F)$ the smallest $C^*$-subalgebra of $\mathcal{C}$ that contains $F$. It corresponds to the intersection of all $C^*$-subalgebras of $\mathcal{C}$ that contains $F$.

**Exercise 1.1.6.** *(i) Show that a $*$-homomorphism $\varphi$ between $C^*$-algebras is isometric if and only if $\varphi$ is injective.*

*(ii) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism between two $C^*$-algebras, show that the kernel $\mathsf{Ker}(\varphi)$ of $\varphi$ is a $C^*$-subalgebra of $\mathcal{C}$ and that the image $\mathsf{Ran}(\varphi)$ of $\varphi$ is a $C^*$-subalgebra of $\mathcal{Q}$.*

An important result about $C^*$-algebras states that each of them can be represented faithfully in a Hilbert space. More precisely:

**Theorem 1.1.7** (Gelfand-Naimark-Segal (GNS) representation)**.** *For any $C^*$-algebra $\mathcal{C}$ there exists a Hilbert space $\mathcal{H}$ and an injective $*$-homomorphism from $\mathcal{C}$ to $\mathcal{B}(\mathcal{H})$. In other words, every $C^*$-algebra $\mathcal{C}$ is $*$-isomorphic*[2] *to a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$.*

**Extension 1.1.8.** *The proof of this theorem is based on the notion of* states *(positive linear functionals) on a $C^*$-algebra, and on the existence of sufficiently many such states. The construction is rather explicit and can be studied, see for example [Mur90, Thm. 3.4.1].*

The next definition of an ideal is the most suitable one in the context of $C^*$-algebra.

**Definition 1.1.9.** An ideal *in a $C^*$-algebra $\mathcal{C}$ is a (non-trivial) $C^*$-subalgebra $\mathcal{J}$ of $\mathcal{C}$ such that $ab \in \mathcal{J}$ and $ba \in \mathcal{J}$ whenever $a \in \mathcal{J}$ and $b \in \mathcal{C}$. This ideal $\mathcal{J}$ is said to be* maximal *in $\mathcal{C}$ if $\mathcal{J}$ is proper ($\Leftrightarrow$ not equal to $\mathcal{C}$) and if $\mathcal{J}$ is not contained in any other proper ideal of $\mathcal{C}$.*

For example, $C_0(\Omega)$ is an ideal of $C_b(\Omega)$, while $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$. Let us add one more important result about the quotient of a $C^*$-algebra by any of its ideals. In this setting we set

$$\mathcal{C}/\mathcal{J} = \{a + \mathcal{J} \mid a \in \mathcal{C}\} \quad \text{and} \quad \|a + \mathcal{J}\| := \inf_{b \in \mathcal{J}} \|a + b\|.$$

In this way $\mathcal{C}/\mathcal{J}$ becomes a $C^*$-algebra, and if one sets $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ by $\pi(a) = a + \mathcal{J}$, then $\pi$ is a $*$-homomorphism with $\mathcal{J} = \mathsf{Ker}(\pi)$. The $*$-homomorphism $\pi$ is called *the quotient map*. We refer to [Mur90, Thm. 3.1.4] for the proof about the quotient $\mathcal{C}/\mathcal{J}$.

Consider now a (finite or infinite) sequence of $C^*$-algebras and $*$-homomorphisms

$$\cdots \longrightarrow \mathcal{C}_n \xrightarrow{\varphi_n} \mathcal{C}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{C}_{n+2} \longrightarrow \dots$$

[2] A $*$-isomorphism is a bijective $*$-homomorphism.

This sequence is *exact* if $\mathsf{Ran}(\varphi_n) = \mathsf{Ker}(\varphi_{n+1})$ for any $n$. A sequence of the form

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0 \tag{1.1}$$

is called *a short exact sequence*. In particular, if $\mathcal{J}$ is an ideal in $\mathcal{C}$ we can consider

$$0 \longrightarrow \mathcal{J} \xhookrightarrow{\iota} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{J} \longrightarrow 0$$

where $\iota$ is the inclusion map and $\pi$ the quotient map already introduced.

If in (1.1) there exists a $*$-homomorphism $\lambda : \mathcal{Q} \to \mathcal{C}$ such that $\psi \circ \lambda = \mathrm{id}$, then $\lambda$ is *a lift for* $\psi$, and the short exact sequence is said to be *split exact*. For example, let $\mathcal{C}_1, \mathcal{C}_2$ be $C^*$-algebras, and consider the direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ with the pointwise multiplication and involution, and the supremum norm. One can then observe that the following short exact sequence

$$0 \longrightarrow \mathcal{C}_1 \xrightarrow{\iota_1} \mathcal{C}_1 \oplus \mathcal{C}_2 \xrightarrow{\pi_2} \mathcal{C}_2 \longrightarrow 0$$

is split exact, when $\iota_1$ and $\pi_2$ are defined by $\iota_1(a) = (a, 0)$ and $\pi_2(a, b) = b$. Indeed, one can set $\lambda : \mathcal{C}_2 \to \mathcal{C}_1 \oplus \mathcal{C}_2$ with $\lambda(b) = (0, b)$ and the equality $\pi_2 \circ \lambda = \mathrm{id}$ holds. Note that neither all short exact sequences are split exact, nor all split exact short exact sequences are direct sums.

Let us finally mention that with any $C^*$-algebra $\mathcal{C}$ one can associate a unique unital $C^*$-algebra $\widetilde{\mathcal{C}}$ which contains $\mathcal{C}$ as an ideal and such that $\widetilde{\mathcal{C}}/\mathcal{C} = \mathbb{C}$. In addition, the short exact sequence

$$0 \longrightarrow \mathcal{C} \xhookrightarrow{\iota} \widetilde{\mathcal{C}} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

is split exact, with $\lambda(\alpha) = \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$. Here $\mathbf{1}$ denotes the identity element of $\widetilde{\mathcal{C}}$. The $C^*$-algebra $\widetilde{\mathcal{C}}$ is called *the (smallest) unitization of* $\mathcal{C}$. Note that

$$\widetilde{\mathcal{C}} = \big\{a + \alpha\mathbf{1} \mid a \in \mathcal{C}, \alpha \in \mathbb{C}\big\}, \tag{1.2}$$

and therefore $\mathcal{C}$ is naturally identified with the element of the form $a + 0\mathbf{1}$ in $\widetilde{\mathcal{C}}$.

**Exercise 1.1.10.** *Work out the details of the construction of $\widetilde{\mathcal{C}}$, see for example [RLL00, Exercise 1.3].*

An important property of the previous construction is its functoriality, in the sense that for any $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between $C^*$-algebras, there exists a unique unit preserving $*$-homomorphism $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ such that $\tilde{\varphi} \circ \iota_{\mathcal{C}} = \iota_{\mathcal{Q}} \circ \varphi$. This morphism is defined by $\tilde{\varphi}(a + \alpha\mathbf{1}_{\widetilde{\mathcal{C}}}) = \varphi(a) + \alpha\mathbf{1}_{\widetilde{\mathcal{Q}}}$ for any $a \in \mathcal{C}$ and $\alpha \in \mathbb{C}$.

## 1.2 Spectral theory

Let us now consider an arbitrary unital $C^*$-algebra $\mathcal{C}$, and let $a \in \mathcal{C}$. One says that $a$ is *invertible* if there exists $b \in \mathcal{C}$ such that $ab = \mathbf{1} = ba$. In this case, the element $b$ is denoted by $a^{-1}$ and is called *the inverse of* $a$. The set of all invertible elements is denoted by $\mathcal{GL}(\mathcal{C})$. Clearly, $\mathcal{GL}(\mathcal{C})$ is a group.

**Exercise 1.2.1.** *Show that $\mathcal{GL}(\mathcal{C})$ is an open set in any unital $C^*$-algebra $\mathcal{C}$, and that the map $\mathcal{GL}(\mathcal{C}) \ni a \mapsto a^{-1} \in \mathcal{C}$ is differentiable. The Neumann series can be used in the proof, namely if $\|a\| < 1$ one has*

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n. \tag{1.3}$$

Note that in the sequel, we shall sometimes write $a - z$ for $a - z\mathbf{1}$, whenever $a$ is an element of a unital $C^*$-algebra and $z \in \mathbb{C}$.

**Definition 1.2.2.** *Let $\mathcal{C}$ be a unital $C^*$-algebra and let $a \in \mathcal{C}$.* The spectrum $\sigma_{\mathcal{C}}(a)$ of $a$ with respect to $\mathcal{C}$ *is defined by*

$$\sigma_{\mathcal{C}}(a) := \big\{z \in \mathbb{C} \mid (a - z\mathbf{1}) \notin \mathcal{GL}(\mathcal{C})\big\}.$$

The spectral radius $r(a)$ of $a$ with respect to $\mathcal{C}$ *is defined by*

$$r(a) := \sup\big\{|z| \mid z \in \sigma_{\mathcal{C}}(a)\big\}.$$

Note that the spectrum $\sigma_{\mathcal{C}}(a)$ of $a$ is a closed subset of $\mathbb{C}$ which is never empty. This result is not completely trivial and its proof is based on Liouville's Theorem in complex analysis. In addition, note that the estimate $r(a) \leq \|a\|$ and the equality $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ always hold. We refer to [Mur90, Sec. 1.2] for the proofs of these statements. Let us mention that if $\mathcal{C}$ has no unit, the spectrum of an element $a \in \mathcal{C}$ can still be defined by $\sigma_{\mathcal{C}}(a) := \sigma_{\tilde{\mathcal{C}}}(a)$.

Based on these observations, we state two results which are often quite useful.

**Theorem 1.2.3** (Gelfand-Mazur)**.** *If $\mathcal{C}$ is a unital $C^*$-algebra in which every non-zero element is invertible, then $\mathcal{C} = \mathbb{C}\mathbf{1}$.*

*Proof.* We know from the observation made above that for any $a \in \mathcal{C}$, there exists $z \in \mathbb{C}$ such that $a - z\mathbf{1} \notin \mathcal{GL}(\mathcal{C})$. By assumption, it follows that $a - z\mathbf{1} = 0$, which means $a = z\mathbf{1}$. □

**Lemma 1.2.4.** *Let $\mathcal{J}$ be a maximal ideal of a unital Abelian $C^*$-algebra $\mathcal{C}$, then $\mathcal{C}/\mathcal{J} = \mathbb{C}\mathbf{1}$.*

*Proof.* As already mentioned, $\mathcal{C}/\mathcal{J}$ is a $C^*$-algebra with unit $\mathbf{1} + \mathcal{J}$; we denote the quotient map $\mathcal{C} \to \mathcal{C}/\mathcal{J}$ by $\pi$. If $\mathcal{I}$ is an ideal in $\mathcal{C}/\mathcal{J}$, then $\pi^{-1}(\mathcal{I})$ is an ideal of $\mathcal{C}$ containing $\mathcal{J}$, which is therefore either equal to $\mathcal{C}$ or to $\mathcal{J}$, by the maximality of $\mathcal{J}$. Consequently, $\mathcal{I}$ is either equal to $\mathcal{C}/\mathcal{J}$ or to 0, and $\mathcal{C}/\mathcal{J}$ has no proper ideal.

Now, if $a \in \mathcal{C}/\mathcal{J}$ and $a \neq 0$, then $a \in \mathcal{GL}\big(\mathcal{C}/\mathcal{J}\big)$, since otherwise $a(\mathcal{C}/\mathcal{J})$ would be a proper ideal of $\mathcal{C}/\mathcal{J}$. In other words, one has obtained that any non-zero element of $\mathcal{C}/\mathcal{J}$ is invertible, which implies that $\mathcal{C}/\mathcal{J} = \mathbb{C}\mathbf{1}$, by Theorem 1.2.3. □

The following statement is an important result for spectral theory in the framework of $C^*$-algebras. It shows that the computation of the spectrum does not depend on the surrounding algebra.

**Theorem 1.2.5.** *Let $\mathcal{C}$ be a $C^*$-subalgebra of a unital $C^*$-algebra $\mathcal{Q}$ which contains the unit of $\mathcal{Q}$. Then for any $a \in \mathcal{C}$,*

$$\sigma_{\mathcal{C}}(a) = \sigma_{\mathcal{Q}}(a).$$

The proof of this theorem is mainly based on the previous lemmas, but requires some preliminary works. We refer to [Mur90, Thm. 1.2.8 & 2.1.11] for its proof. Note that because of this result, it is common to denote by $\sigma(a)$ the spectrum of an element $a$ of a $C^*$-algebra, without specifying in which algebra the spectrum is computed.

In the next definition we consider some special elements of a $C^*$-algebra.

**Definition 1.2.6.** *Let $\mathcal{C}$ be a $C^*$-algebra and let $a \in \mathcal{C}$. The element $a$ is* self-adjoint *or* hermitian *if $a = a^*$, $a$ is* normal *if $aa^* = a^*a$. If $a$ is self-adjoint and $\sigma(a) \subset \mathbb{R}_+$, then $a$ is said to be positive. If $\mathcal{C}$ is unital and if $u \in \mathcal{C}$ satisfies $uu^* = u^*u = \mathbf{1}$, then $u$ is said to be* unitary.

The set of all positive elements in $\mathcal{C}$ is usually denoted by $\mathcal{C}^+$, and one simply writes $a \geq 0$ to mean that $a$ is positive. An important result in this context is that for any $a \in \mathcal{C}^+$, there exists $b \in \mathcal{C}$ such that $a = b^*b$. One can even strengthen this result by showing that for any $a \in \mathcal{C}^+$, there exists a unique $b \in \mathcal{C}^+$ such that $a = b^2$. This element $b$ is usually denoted by $a^{1/2}$. Now, for any self-adjoint operators $a_1, a_2$, one writes $a_1 \geq a_2$ if $a_1 - a_2 \geq 0$. For completeness, we add some information about $\mathcal{C}^+$.

**Proposition 1.2.7.** *Let $\mathcal{C}$ be a $C^*$-algebra. Then,*

*(i) The sum of two positive elements of $\mathcal{C}$ is a positive element of $\mathcal{C}$,*

*(ii) The set $\mathcal{C}^+$ is equal to $\{a^*a \mid a \in \mathcal{C}\}$,*

*(iii) If $a, b$ are self-adjoint elements of $\mathcal{C}$ and if $c \in \mathcal{C}$, then $a \geq b \Rightarrow c^*ac \geq c^*bc$,*

*(iv) If $a \geq b \geq 0$, then $a^{1/2} \geq b^{1/2}$,*

*(v) If $a \geq b \geq 0$, then $\|a\| \geq \|b\|$,*

*(vi) If $\mathcal{C}$ is unital and $a, b$ are positive and invertible elements of $\mathcal{C}$, then $a \geq b \Rightarrow b^{-1} \geq a^{-1} \geq 0$,*

*(vii) For any $a \in \mathcal{C}$ there exist $a_1, a_2, a_3, a_4 \in \mathcal{C}^+$ such that*

$$a = a_1 - a_2 + ia_3 - ia_4.$$

*Proof.* See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur90]. □

In the next statement, we provide some information on the spectrum of self-adjoint and unitary elements of a unital $C^*$-algebra. For that purpose, we immediately infer from the equality $\|u^*u\| = \|u\|^2$ that if $u$ is unitary, then $\|u\| = 1$. We also set

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

**Lemma 1.2.8.** *Any self-adjoint element $a$ in a unital $C^*$-algebra $\mathcal{C}$ satisfies $\sigma(a) \subset \mathbb{R}$. If $u$ is a unitary element of $\mathcal{C}$, then $\sigma(u) \subset \mathbb{T}$.*

*Proof.* First of all, let $b \in \mathcal{C}$ and observe that from the equality $\big((b-z)^{-1}\big)^* = (b^*-\overline{z})^{-1}$, one infers that if $z \in \sigma(b)$, then $\overline{z} \in \sigma(b^*)$. Furthermore, from the equality

$$z^{-1}(z-b)b^{-1} = -(z^{-1} - b^{-1}),$$

one also deduces that if $z \in \sigma(b)$ for some $b \in \mathcal{GL}(\mathcal{C})$, then $z^{-1} \in \sigma(b^{-1})$.

Now, for a unitary $u \in \mathcal{C}$, one deduces from the above computations that if $z \in \sigma(u)$, then $\overline{z}^{-1} \in \sigma\big((u^*)^{-1}\big) = \sigma(u)$. Since $\|u\| = 1$ one then infers from the equality $r(u) = \|u\| = 1$ that $|z| \leq 1$ and $|z^{-1}| \leq 1$, which means $z \in \mathbb{T}$.

If $a = a^* \in \mathcal{C}$, one sets $e^{ia} := \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$ and observes that

$$(e^{ia})^* = e^{-ia} = (e^{ia})^{-1}.$$

Therefore, $e^{ia}$ is a unitary element of $\mathcal{C}$ and it follows that $\sigma\big(e^{ia}\big) \subset \mathbb{T}$. Now, let us assume that $z \in \sigma(a)$, set $b := \sum_{n=1}^{\infty} \frac{i^n(a-z)^{n-1}}{n!}$, and observe that $b$ commutes with $a$. Then one has

$$e^{ia} - e^{iz} = (e^{i(a-z)} - 1)e^{iz} = (a-z)be^{iz}.$$

It follows from this equality that $e^{iz} \in \sigma(e^{ia})$. Indeed, if $\big(e^{ia} - e^{iz}\big) \in \mathcal{GL}(\mathcal{C})$, then $be^{iz}\big(e^{ia} - e^{iz}\big)^{-1}$ would be an inverse for $(a-z)$, which can not be since $z \in \sigma(a)$. From the preliminary computation, one deduces that $|e^{iz}| = 1$, which holds if and only if $z \in \mathbb{R}$. One has thus obtains that $\sigma(a) \subset \mathbb{R}$. □

Let us now state an important result for Abelian $C^*$-algebras.

**Theorem 1.2.9** (Gelfand)**.** *Any Abelian $C^*$-algebra $\mathcal{C}$ is $*$-isomorphic to a $C^*$-algebra of the form $C_0(\Omega)$ for some locally compact Hausdorff*[3] *space $\Omega$.*

In fact, Gelfand's theorem provides more information, namely

(i) The mentioned $*$-isomorphism is isometric,

(ii) $\Omega$ is compact if and only if $\mathcal{C}$ is unital,

(iii) $\Omega$ and $\Omega'$ are homeomorphic if and only if $C_0(\Omega)$ and $C_0(\Omega')$ are $*$-isomorphic,

(iv) The set $\Omega$ is called *the spectrum* of $\mathcal{C}$ and corresponds to the set of *characters* of $\mathcal{C}$ endowed with a suitable topology. A character on $\mathcal{C}$ is a non-zero $*$-homomorphism from $\mathcal{C}$ to $\mathbb{C}$.

[3] A Hausdorff space is a topological space in which distinct points have disjoint neighbourhoods.

In this context, let us mention that there exists a bijective correspondence between open subsets of $\Omega$ and ideals in $C_0(X)$. For example, if $X$ is any open subset of $\Omega$, then $C_0(X) \subset C_0(\Omega)$ (by extending the element of $C_0(X)$ by 0 on $\Omega \setminus X$) and $C_0(X)$ is then clearly an ideal of $C_0(\Omega)$. As a consequence, one gets the following short exact sequence:

$$0 \longrightarrow C_0(X) \overset{\iota}{\hookrightarrow} C_0(\Omega) \overset{\pi}{\longrightarrow} C_0(\Omega \setminus X) \longrightarrow 0.$$

**Extension 1.2.10.** *Write down the details of the construction of the Gelfand transform, first for Banach algebras, and then for $C^*$-algebras. Provide a proof of the above statements.*

The Gelfand representation has various useful applications. One is contained in the proof of the following statement, see [Mur90, Thm. 2.1.13] for its proof. This statement corresponds to a so-called *bounded functional calculus.*

**Proposition 1.2.11.** *Let $a$ be a normal element of a unital $C^*$-algebra $\mathcal{C}$, and let $\iota : \sigma(a) \to \mathbb{C}$ be the inclusion map,* i.e. *$\iota(z) = z$ for any $z \in \sigma(a)$. Then there exists a unique unital $*$-homomorphism $\varphi_a : C\big(\sigma(a)\big) \to \mathcal{C}$ satisfying $\varphi_a(\iota) = a$. Moreover, $\varphi_a$ is isometric and the image of $\varphi_a$ is the $C^*$-subalgebra $C^*(\{a, \mathbf{1}\})$ of $\mathcal{C}$ generated by $a$ and $\mathbf{1}$.*

Note that if $f$ is a polynomial, then the equality $\varphi_a(f) = f(a)$ holds, and if $f$ corresponds to the map $f(z) = \bar{z}$, then one has $\varphi_a(f) = a^*$. For the former reason, one usually write simply $f(a)$ instead of $\varphi_a(f)$ for any $f \in C\big(\sigma(a)\big)$. We also mention a useful result about the spectrum of elements obtained by the previous bounded functional calculus [Mur90, Thm. 2.1.14].

**Theorem 1.2.12** (Spectral mapping theorem)**.** *Let $a$ be a normal element in a unital $C^*$-algebra $\mathcal{C}$, and let $\varphi_a$ be the $*$-homomorphism mentioned in the previous statement. Then for any $f \in C\big(\sigma(a)\big)$, the following equality holds:*

$$\sigma\big(f(a)\big) = f\big(\sigma(a)\big).$$

Let us still gather some additional spectral properties.

(i) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a unital $*$-homomorphism between unital $C^*$-algebras, and if $a$ is a normal element of $\mathcal{C}$, then $\sigma\big(\varphi(a)\big) \subset \sigma(a)$, or in other words the spectrum of $a$ can not increase through a $*$-homomorphism. In addition, if $f \in C\big(\sigma(a)\big)$, then $f\big(\varphi(a)\big) = \varphi\big(f(a)\big)$.

(ii) If $a$ is a normal element in a non-unital $C^*$-algebra $\mathcal{C}$, then $f(a)$ is *a priori* defined only in its unitization $\widetilde{\mathcal{C}}$. Now, if $\pi : \widetilde{\mathcal{C}} \to \mathbb{C}$ denotes the quotient map and for $a \in \mathcal{C}$, one has by the previous point that

$$\pi\big(f(a)\big) = f\big(\pi(a)\big) = f(0).$$

It thus follows from the description of $\widetilde{\mathcal{C}}$ provided in (1.2) that $f(a)$ belongs to $\mathcal{C}$ if and only if $f(0) = 0$.

(iii) If $a$ is a normal element in a $C^*$-algebra, then $r(a) = \|a\|$.

We finally state a technical result which will be used at several occasions in the next chapter.

**Lemma 1.2.13.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, let $K$ be a non-empty compact subset of $\mathbb{R}$ and let $F_K$ be the set of self-adjoint elements of $\mathcal{C}$ with spectrum in $K$. Then for any fixed $f \in C(K)$, the map*

$$F_k \ni a \mapsto f(a) \in \mathcal{C}$$

*is continuous.*

The proof of this statement is provided in [RLL00, Lem. 1.2.5] and relies on an $\varepsilon/3$-argument.

## 1.3 Matrix algebras

For any $C^*$-algebra $\mathcal{C}$, let us denote by $M_n(\mathcal{C})$ the set of all $n \times n$ matrices with entries in $\mathcal{C}$. Addition, multiplication and involution for such matrices are mimicked from the scalar case, *i.e.* when $\mathcal{C} = \mathbb{C}$. In order to define a $C^*$-norm on $M_n(\mathcal{C})$, let us consider any injective $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and extend this morphism to a $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to \mathcal{B}(\mathcal{H}^n)$ by defining[4]

$$\varphi \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \varphi(a_{11})f_1 + \dots + \varphi(a_{1n})f_n \\ \vdots \\ \varphi(a_{n1})f_1 + \dots + \varphi(a_{nn})f_n \end{pmatrix}$$

for any ${}^t(f_1, \dots, f_n) \in \mathcal{H}^n$ (the notation ${}^t(\dots)$ means the transpose of a vector). Then a $C^*$-norm on $M_n(\mathcal{C})$ is obtained by setting $\|a\| := \|\varphi(a)\|$ for any $a \in M_n(\mathcal{C})$, and this norm is independent of the choice of $\varphi$. Note that the following inequalities hold:

$$\max_{i,j} \|a_{ij}\| \leq \left\| \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \right\| \leq \sum_{i,j} \|a_{ij}\|. \tag{1.4}$$

These inequalities have a useful application. It shows that if $\Omega$ is a topological space and if $f : \Omega \to M_n(\mathcal{C})$, then $f$ is continuous if and only if each function $f_{ij} : \Omega \to \mathcal{C}$ is continuous.

[4]The use of the same notation for the maps $\varphi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ and $\varphi : M_n(\mathcal{C}) \to \mathcal{B}(\mathcal{H}^n)$ is done on purpose. Some authors would use $\varphi_n$ for the second map, but the omission of the index $n$ does not lead to any confusion and simplifies the notation.

Finally, let us mention that if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism between two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, then the map $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ defined by

$$\varphi \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{pmatrix} \tag{1.5}$$

is a $*$-homomorphism, for any $n \in \mathbb{N}^*$. Note that again we have used the same notation for two related but different maps.

# Chapter 2

# Projections and unitary elements

The $K$-theory of a $C^*$-algebra is constructed from equivalence classes of its projections and from equivalence classes of its unitary elements. For that reason, we shall consider several equivalence relations and look at the relations between them. The $K$-groups will be defined only in the following chapters. This chapter is mainly based on Chapter 2 of the book [RLL00].

## 2.1 Homotopy classes of unitary elements

**Definition 2.1.1.** *For any topological space $\Omega$, one says that $a, b \in \Omega$ are* homotopic in $\Omega$ *if there exists a continuous map $v : [0,1] \ni t \mapsto v(t) \in \Omega$ with $v(0) = a$ and $v(1) = b$. In such a case one writes $a \sim_h b$ in $\Omega$.*

Clearly, the relation $\sim_h$ defines an equivalence relation on $\Omega$, and one says that $v$ is a continuous path from $a$ to $b$ in $\Omega$. Note that if $\Omega'$ is another topological space with $a, b \in \Omega'$ as well, then $a, b$ could be homotopic in $\Omega$ without being homotopic in $\Omega'$. Thus, mentioning the ambient space $\Omega$ is crucial for the definition of the homotopy relation. On the other hand, we shall often just write $t \mapsto v(t)$ for the continuous path, without specifying $t \in [0,1]$.

In the next statement, we consider this equivalence relation in the set $\mathcal{U}(\mathcal{C})$ of all unitary elements of a unital $C^*$-algebra $\mathcal{C}$. Clearly, this set is a group (for the multiplication) but not a vector space. Note also that if $u_0, u_1, v_0, v_1 \in \mathcal{U}(\mathcal{C})$ satisfy $u_0 \sim_h u_1$ and $v_0 \sim_h v_1$, then $u_0 v_0 \sim_h u_1 v_1$. Indeed, if $t \mapsto u(t)$ and $t \mapsto v(t)$ denote the corresponding continuous paths, then $t \mapsto u(t)v(t)$ is a continuous map between $u_0 v_0$ and $u_1 v_1$. In the sequel we shall denote by $\mathcal{U}_0(\mathcal{C})$ the set of elements in $\mathcal{U}(\mathcal{C})$ which are homotopic to $\mathbf{1} \in \mathcal{C}$. Let us also recall from Lemma 1.2.8 that for any unitary element $u$, one has $\sigma(u) \in \mathbb{T}$.

**Lemma 2.1.2.** *Let $\mathcal{C}$ be a unital $C^*$-algebra. Then:*

*(i) If $a \in \mathcal{C}$ is self-adjoint, then $e^{ia}$ belongs to $\mathcal{U}_0(\mathcal{C})$,*

*(ii) If $u \in \mathcal{C}$ is unitary and $\sigma(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(\mathcal{C})$,*

*(iii) If $u, v \in \mathcal{C}$ are unitary with $\|u - v\| < 2$, then $u \sim_h v$.*

*Proof.* i) In the proof of Lemma 1.2.8 it has already been observed that if $a$ is self-adjoint, then $e^{ia}$ is unitary. By considering now the map $[0,1] \ni t \mapsto v(t) := e^{ita} \in \mathcal{U}(\mathcal{C})$, one easily observes that this map is continuous, and that $v(0) = \mathbf{1}$ and $v(1) = e^{ia}$. As a consequence, $e^{ia} \sim_h \mathbf{1}$, or equivalently $e^{ia} \in \mathcal{U}_0(\mathcal{C})$.

ii) Since $\sigma(u) \neq \mathbb{T}$, there exists $\theta \in \mathbb{R}$ such that $e^{i\theta} \notin \sigma(u)$. Let us then define $v : \sigma(u) \to \mathbb{R}$ by $v\big(e^{i(\theta+t)}\big) = \theta + t$ for any $t \in (0, 2\pi)$ such that $e^{i(\theta+t)} \in \sigma(u)$. Since $\sigma(u)$ is a closed set in $\mathbb{T}$, it follows that $v$ is continuous. In addition, one has that $e^{iv(z)} = z$ for any $z \in \sigma(u)$. Thus, if one sets $a := v(u)$, one infers that $a$ is a self-adjoint element of $\mathcal{C}$ and that $u = e^{ia}$. As a consequence of $(i)$, one deduces that $u \in \mathcal{U}_0(\mathcal{C})$.

iii) If $\|u - v\| < 2$, it follows that $\|v^*u - \mathbf{1}\| = \|v^*(u - v)\| < 2$. Then, from the estimates $|z| \leq r(a) \leq \|a\|$ valid for any $z \in \sigma(a)$ and any $a \in \mathcal{C}$, one infers that $-2 \notin \sigma(v^*u - \mathbf{1})$, or equivalently $-1 \notin \sigma(v^*u)$. Since $v^*u$ is a unitary element of $\mathcal{C}$, one infers then from $(ii)$ that $v^*u \sim_h \mathbf{1}$. Finally, by multiplying the corresponding continuous path on the left by $v$ (or by using the remark made just before the statement of the lemma), one infers that $u \sim_h v$, as expected. □

Let us stress that the previous lemma states that for any self-adjoint $a \in \mathcal{C}$, $e^{ia}$ is a unitary element of $\mathcal{U}_0(\mathcal{C})$. However, not all unitary elements of $\mathcal{C}$ are of this form, and the point $(ii)$ has only provided a sufficient condition for being of this form. Later on, we shall construct unitary elements which are not obtained from a self-adjoint element.

Let us observe that since unitary elements of $M_n(\mathbb{C})$ have only a finite spectrum, one can directly infer from the previous statement $(ii)$ the following corollary:

**Corollary 2.1.3.** *The unitary group in $M_n(\mathbb{C})$ is connected, or in other words*

$$\mathcal{U}_0\big(M_n(\mathbb{C})\big) = \mathcal{U}\big(M_n(\mathbb{C})\big).$$

By considering matrix algebras, the following statement can easily be proved:

**Lemma 2.1.4** (Whitehead)**.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, and let $u, v \in \mathcal{U}(\mathcal{C})$. Then one has in $\mathcal{U}\big(M_2(\mathcal{C})\big)$*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

*In particular, one infers that*

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{2.1}$$

*in $\mathcal{U}\big(M_2(\mathcal{C})\big)$.*

*Proof.* Since $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$ is a unitary element of $M_2(\mathcal{C})$, one infers from the previous corollary that $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$. Then, by observing that

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

one readily infers that the r.h.s. is homotopic to $\left(\begin{smallmatrix} u & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} v & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} uv & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$. The other relations can be proved similarly. □

Let us add some more information on $\mathcal{U}_0(\mathcal{C})$.

**Proposition 2.1.5.** *Let $\mathcal{C}$ be a unital $C^*$-algebra. Then,*

*(i) $\mathcal{U}_0(\mathcal{C})$ is a normal subgroup of $\mathcal{U}(\mathcal{C})$,* i.e. *$vuv^* \in \mathcal{U}_0(\mathcal{C})$ whenever $u \in \mathcal{U}_0(\mathcal{C})$ and $v \in \mathcal{U}(\mathcal{C})$,*

*(ii) $\mathcal{U}_0(\mathcal{C})$ is open and closed relative to $\mathcal{U}(\mathcal{C})$,*

*(iii) An element $u \in \mathcal{C}$ belongs to $\mathcal{U}_0(\mathcal{C})$ if and only if $u = e^{ia_1}\, e^{ia_2} \dots e^{ia_n}$ for some self-adjoint elements $a_1, \dots, a_n \in \mathcal{C}$.*

**Exercise 2.1.6.** *Provide the proof of this statement, see also [RLL00, Prop. 2.1.6].*

Based on the content of this proposition, the following lemma can be proved:

**Lemma 2.1.7.** *Let $\mathcal{C}$, $\mathcal{Q}$ be unital $C^*$-algebras, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective (and hence unit preserving) $*$-homomorphism. Then:*

*(i) $\varphi\big(\mathcal{U}_0(\mathcal{C})\big) = \mathcal{U}_0(\mathcal{Q})$,*

*(ii) For each $u \in \mathcal{U}(\mathcal{Q})$ there exists $v \in \mathcal{U}_0\big(M_2(\mathcal{C})\big)$ such that $\varphi(v) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$,*

*(iii) If $u \in \mathcal{U}(\mathcal{Q})$, and if there exists $v \in \mathcal{U}(\mathcal{C})$ such that $u \sim_h \varphi(v)$, then $u$ belongs to $\varphi\big(\mathcal{U}(\mathcal{C})\big)$.*

*Proof.* i) Since a unital $*$-homomorphism is continuous and maps unitary elements on unitary elements, it follows that $\varphi\big(\mathcal{U}_0(\mathcal{C})\big)$ is contained in $\mathcal{U}_0(\mathcal{Q})$. Conversely, if $u$ belongs to $\mathcal{U}_0(\mathcal{Q})$, then $u = e^{ib_1}\, e^{ib_2} \dots e^{ib_n}$ for some self-adjoint elements $b_1, \dots, b_n \in \mathcal{Q}$ by Proposition 2.1.5.(*iii*). Since $\varphi$ is surjective, there exists $a_j \in \mathcal{C}$ such that $b_j = \varphi(a_j)$ for any $j \in \{1, \dots, n\}$. Note that $a_j$ can be chosen self-adjoint since otherwise the element $(a_j + a_j^*)/2$ is self-adjoint and satisfies $\varphi\big((a_j + a_j^*)/2\big) = (b_j + b_j^*)/2 = b_j$. Then, by setting $v = e^{ia_1}\, e^{ia_2} \dots e^{ia_n}$ one gets, again by Proposition 2.1.5, that $v \in \mathcal{U}_0(\mathcal{C})$ and that $\varphi(v) = u$.

ii) For any $u \in \mathcal{U}(\mathcal{Q})$ consider the element $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$ which belongs to $\mathcal{U}_0\big(M_2(\mathcal{Q})\big)$ by (2.1). By applying then the point (*i*) to $\mathcal{U}_0\big(M_2(\mathcal{C})\big)$ and $\mathcal{U}_0\big(M_2(\mathcal{Q})\big)$ instead of $\mathcal{U}_0(\mathcal{C})$ and $\mathcal{U}_0(\mathcal{Q})$, one immediately deduces the second statement.

iii) If $u \sim_h \varphi(v)$, then $u\varphi(v)^* = u\varphi(v^*)$ is homotopic to $\mathbf{1} \in \mathcal{Q}$, *i.e.* $u\varphi(v^*) \in \mathcal{U}_0(\mathcal{Q})$. By (*i*) it follows that $u\varphi(v^*) = \varphi(w)$ for some $w \in \mathcal{U}_0(\mathcal{C})$. Consequently, one infers that $u = \varphi(wv)$, or in other words $u \in \varphi\big(\mathcal{U}(\mathcal{C})\big)$. □

Recall that for any unital $C^*$-algebra $\mathcal{C}$, one denotes by $\mathcal{GL}(\mathcal{C})$ the group of its invertible elements. The set of elements of $\mathcal{GL}(\mathcal{C})$ which are homotopic to $\mathbf{1}$ is denoted by $\mathcal{GL}_0(\mathcal{C})$. Clearly, $\mathcal{U}(\mathcal{C})$ is a subgroup of $\mathcal{GL}(\mathcal{C})$. The following statement establishes a more precise link between these two groups. Before this, observe that for any $a \in \mathcal{C}$, the element $a^*a$ is positive, as recalled in Proposition 1.2.7. Thus, one can define $|a| := (a^*a)^{1/2}$ which is also a positive element of $\mathcal{C}$, and call it *the absolute value of* $a$.

**Proposition 2.1.8.** *Let $\mathcal{C}$ be a unital $C^*$-algebra.*

*(i) If $a$ belongs to $\mathcal{GL}(\mathcal{C})$, then $|a|$ belongs to $\mathcal{GL}(\mathcal{C})$ as well, and $w(a) := a|a|^{-1}$ belongs to $\mathcal{U}(\mathcal{C})$. In addition, the equality $a = w(a)|a|$ holds.*

*(ii) The map*

$$w : \mathcal{GL}(\mathcal{C}) \ni a \mapsto w(a) \in \mathcal{U}(\mathcal{C})$$

*is continuous, satisfies $w(u) = u$ for any $u \in \mathcal{U}(\mathcal{C})$, and verifies $w(a) \sim_h a$ in $\mathcal{GL}(\mathcal{C})$ for any $a \in \mathcal{GL}(\mathcal{C})$,*

*(iii) If $v_0, v_1 \in \mathcal{U}(\mathcal{C})$ satisfies $v_0 \sim_h v_1$ in $\mathcal{GL}(\mathcal{C})$, then $v_0 \sim_h v_1$ in $\mathcal{U}(\mathcal{C})$.*

*Proof.* i) If $a$ is invertible, it follows that $a^*$ and $a^*a$ are invertible as well. As a consequence, the element $|a| = (a^*a)^{1/2}$ is also invertible, with inverse $\big((a^*a)^{-1}\big)^{1/2}$. For simplicity, let us set $w := a|a|^{-1}$ which verifies $a = w|a|$. Since $w$ is the product of two invertible elements, $w$ is invertible as well, and it satisfies $w^* = w^{-1}$ since

$$w^*w = |a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = \mathbf{1}.$$

Consequently, $w \in \mathcal{U}(\mathcal{C})$.

ii) The continuity of the map $a \mapsto a^{-1}$ in $\mathcal{GL}(\mathcal{C})$ can easily be obtained by the Neumann series, as recalled in Exercise 1.2.1. Thus, to show that the map $a \mapsto w(a)$ is continuous, it is sufficient to show that the map $a \mapsto (a^*a)^{1/2}$ is continuous. Clearly, the map $a \mapsto a^*a$ is continuous, because involution and multiplication are continuous. It remains to show the continuity of the map $b \mapsto b^{1/2}$ on any bounded subset $F$ of $\mathcal{C}^+$. However, this directly follows from Lemma 1.2.13 since any bounded subset $F$ of $\mathcal{C}^+$ is contained in some $F_K$ (in the notation of the mentioned lemma) with $K = [0, R]$ and $R := \sup\{\|a\| \mid a \in F\}$.

If $u$ is unitary, one has $u^*u = \mathbf{1}$ and thus $|u| = \mathbf{1}$, which implies that $w(u) = u$. On the other hand, for $a \in \mathcal{GL}(\mathcal{C})$, let us set $v(t) = w(a)\big(t|a| + (1-t)\mathbf{1}\big)$ with $t \in [0,1]$. Clearly, $v(0) = w(a)$ and $v(1) = a$, and let us show $v(t) \in \mathcal{GL}(\mathcal{C})$ for any $t$. Indeed, since $|a|$ is positive and invertible, it follows that $\lambda := \inf \sigma(|a|) > 0$, from which one infers that $t|a| + (1-t)\mathbf{1} \geq \min\{\lambda, 1\}\mathbf{1} > 0$. As a consequence of Proposition 1.2.7.(*vi*), it follows that $t|a| + (1-t)\mathbf{1}$ is invertible, and therefore $v(t)$ is invertible as well. Since the map $t \mapsto v(t)$ is continuous, one concludes that $w(a) \sim_h a$ in $\mathcal{GL}(\mathcal{C})$.

iii) If $t \mapsto v(t)$ is a continuous path in $\mathcal{GL}(\mathcal{C})$ between $v_0$ and $v_1$, then $t \mapsto w\big(v(t)\big)$ is a continuous path in $\mathcal{U}(\mathcal{C})$ between $v_0$ and $v_1$. $\square$

The above proposition says that $\mathcal{U}(\mathcal{C})$ is *a retract*[1] *of* $\mathcal{GL}(\mathcal{C})$. Note also that the above decomposition $a = w(a)|a|$ for any invertible element $a$ of $\mathcal{C}$ is called *the polar decomposition of* $a$. This decomposition is often written $a = u|a|$ with $u := w(a)$.

Finally, let us state a useful result:

**Lemma 2.1.9.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, and let $a \in \mathcal{C}$ be invertible. Assume that $b \in \mathcal{C}$ satisfies $\|b - a\| < \|a^{-1}\|^{-1}$. Then $b$ is invertible, with*

$$\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\|,$$

*and $a \sim_h b$ in $\mathcal{GL}(\mathcal{C})$.*

**Exercise 2.1.10.** *Provide a proof of the previous lemma, with the possible help of [RLL00, Prop. 2.1.11].*

## 2.2 Equivalence of projections

We start with the definition of a (self-adjoint) projection in the setting of a $C^*$-algebra.

**Definition 2.2.1.** *An element $p$ in a $C^*$-algebra $\mathcal{C}$ is called* a projection *if $p = p^2 = p^*$. The set of all projections in $\mathcal{C}$ is denoted by $\mathcal{P}(\mathcal{C})$.*

**Exercise 2.2.2.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, and let $p \in \mathcal{P}(\mathcal{C})$. Show that $\sigma(p) \subset \{0, 1\}$.*

Clearly, the equivalence by homotopy $\sim_h$ can be considered on $\mathcal{P}(\mathcal{C})$, but let us consider two additional equivalence relations. Namely, for any $p, q \in \mathcal{P}(\mathcal{C})$, one writes $p \sim q$ if there exists $v \in \mathcal{C}$ such that $p = v^*v$ and $q = vv^*$ and calls it *the Murray-von Neumann equivalence*. Alternatively, one writes $p \sim_u q$ if there exists an element $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$ and calls it *the unitary equivalence*. Note that an element $v$ of $\mathcal{C}$ satisfying $v^*v, vv^* \in \mathcal{P}(\mathcal{C})$ is called *a partial isometry*. The projection $p := v^*v$ is called *the support projection of* $v$, and the projection $q := vv^*$ is called *the range projection of* $v$. We can then observe that in this setting one has

$$v = qv = vp = qvp. \tag{2.2}$$

**Exercise 2.2.3.** *Show that for any $v$ in a $C^*$-algebra such that $v^*v$ is a projection, then automatically $vv^*$ is also a projection. By using the equalities provided in* (2.2)*, show that the Murray-von Neumann relation is transitive.*

**Lemma 2.2.4.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, and let $p, q \in \mathcal{P}(\mathcal{C})$. Then the following statements are equivalent:*

*(i)* $p \sim_u q$,

[1] A *retract* of a topological space $\Omega$ consists in a subspace $\Omega_0$ such that there exists a continuous map $\tau : \Omega \to \Omega_0$ satisfying $x \sim_h \tau(x)$ in $\Omega$, for any $x \in \Omega$, and such that $\tau(x) = x$ for all $x \in \Omega_0$.

*(ii)* $q = upu^*$ *for some* $u \in \mathcal{U}(\mathcal{C})$,

*(iii)* $p \sim q$ *and* $\mathbf{1} - p \sim \mathbf{1} - q$.

*Proof.* Let us denote by $\tilde{\mathbf{1}}$ the unit of $\widetilde{\mathcal{C}}$ and keep the notation $\mathbf{1}$ for the unit of $\mathcal{C}$. We set $1 := \tilde{\mathbf{1}} - \mathbf{1}$, and one can observe that 1 is a projection in $\widetilde{\mathcal{C}}$. In addition, one has

$$\widetilde{\mathcal{C}} = \big\{a + \alpha 1 \mid a \in \mathcal{C}, \alpha \in \mathbb{C}\big\}$$

and $a1 = 1a = 0$ for any $a \in \mathcal{C}$.

$(i) \Rightarrow (ii)$: Assume that $q = vpv^*$ for some $v \in \mathcal{U}(\widetilde{\mathcal{C}})$. By the previous observation, one has $v = u + \alpha 1$ for some $u \in \mathcal{C}$ and $\alpha \in \mathbb{C}$. By computing $v^*v$ and $vv^*$, one readily infers that $u \in \mathcal{U}(\mathcal{C})$ and then that $q = upu^*$.

$(ii) \Rightarrow (iii)$: Suppose that $q = upu^*$ for some $u \in \mathcal{U}(\mathcal{C})$. By setting $v := up$ and $w := u(\mathbf{1} - p)$ one gets

$$v^*v = p, \quad vv^* = q, \quad w^*w = \mathbf{1} - p, \quad ww^* = \mathbf{1} - q. \tag{2.3}$$

$(iii) \Rightarrow (i)$: Suppose that there are partial isometries $v$ and $w$ in $\mathcal{C}$ satisfying (2.3). By setting $u := v + w + 1$ and by taking (2.3) and the definition of 1 into account one gets

$$uu^* = vv^* + ww^* + wv^* + vw^* + (\tilde{\mathbf{1}} - \mathbf{1}) = wv^* + vw^* + \tilde{\mathbf{1}}$$

and

$$u^*u = v^*v + w^*w + w^*v + v^*w + (\tilde{\mathbf{1}} - \mathbf{1}) = w^*v + v^*w + \tilde{\mathbf{1}}.$$

Then, by inserting the support and the range projections one readily obtains $wv^* = w(1-p)pv^* = 0$, and similarly $vw^* = 0$, $w^*v = 0$ and $v^*w = 0$, which imply that $u \in \mathcal{U}(\widetilde{\mathcal{C}})$. We finally find that $upu^* = vpv^* = vv^* = q$, as expected. □

Let us now state a short technical result, which proof can be found in [RLL00, Lem. 2.2.3].

**Lemma 2.2.5.** *Let* $\mathcal{C}$ *be a* $C^*$*-algebra and let* $p \in \mathcal{P}(\mathcal{C})$ *and* $a \in \mathcal{C}$ *be self-adjoint. By setting* $\delta := \|p - a\|$, *one has*

$$\sigma(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

Based on the previous lemma, one can now show the following statement:

**Proposition 2.2.6.** *Let* $\mathcal{C}$ *be a* $C^*$*-algebra, and let* $p, q \in \mathcal{P}(\mathcal{C})$ *with* $\|p - q\| < 1$. *Then* $p \sim_h q$ *in* $\mathcal{P}(\mathcal{C})$.

*Proof.* For any $t \in [0, 1]$, let us set $a(t) := (1 - t)p + tq$. Clearly, $a(t)$ is self-adjoint, it satisfies

$$\min\big\{\|a(t) - p\|, \|a(t) - q\|\big\} \leq \|p - q\|/2 < 1/2,$$

and the map $t \mapsto a(t)$ is continuous. Moreover, by Lemma 2.2.5 and with the notation of Lemma 1.2.13, each $a(t)$ belongs to $F_K$ with $K := [-\delta, \delta] \cup [1-\delta, 1+\delta]$ and $\delta = \|p-q\|/2$. Note that since $\|p-q\| < 1$, these two intervals are disjoint.

Now, let $f$ be the continuous function on $K$ given by $f(x) = 0$ if $x \in [-\delta, \delta]$ and $f(x) = 1$ if $x \in [1-\delta, 1+\delta]$. Then, since $f = f^2 = \overline{f}$, it follows that $f\big(a(t)\big)$ is a projection for each $t \in [0,1]$. In addition, the map $t \mapsto f\big(a(t)\big) \in \mathcal{P}(\mathcal{C})$ is continuous by Lemma 1.2.13, and hence one has in $\mathcal{P}(\mathcal{C})$

$$p = f(p) = f\big(a(0)\big) \sim_h f\big(a(1)\big) = f(q) = q.$$

□

One usually says that two elements $a, b$ in a unital $C^*$-algebra are *similar* if there exists $c \in \mathcal{GL}(\mathcal{C})$ such that $b = cac^{-1}$. In the next statement, we show that if two self-adjoint elements are similar, then they are unitarily equivalent.

**Proposition 2.2.7.** *Let $a, b$ be self-adjoint elements in a unital $C^*$-algebra $\mathcal{C}$, and suppose that there exists $c \in \mathcal{GL}(\mathcal{C})$ such that $b = cac^{-1}$. Let $c = u|c|$ be the polar decomposition of $c$, with $u \in \mathcal{U}(\mathcal{C})$. Then $b = uau^*$.*

*Proof.* Since $a$ and $b$ are self-adjoint, the equation $b = cac^{-1}$ implies that $bc = ca$ and that $ac^* = c^*b$. As a consequence, one infers that

$$|c|^2 a = c^*ca = c^*bc = ac^*c = a|c|^2,$$

which means that $a$ and $|c|^2$ commute. One then deduces that $a$ commutes with all elements of $C^*\big(\{|c|^2, \mathbf{1}\}\big)$ and in particular $a$ commutes with $|c|^{-1}$ (which exists since $c$ is invertible). It thus follows that

$$uau^* = c|c|^{-1}au^* = ca|c|^{-1}u^* = bc|c|^{-1}u^* = buu^* = b.$$

□

Let us add one more information on the relation between $\sim_h$ and the unitary equivalence.

**Proposition 2.2.8.** *Let $\mathcal{C}$ be a $C^*$-algebra, and let $p, q \in \mathcal{P}(\mathcal{C})$. Then $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$ if and only if there exists a unitary element $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$ such that $q = upu^*$.*

*Proof.* Let $\mathbf{1}$ denote the unit of $\widetilde{\mathcal{C}}$, and assume that there exists $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$ which verifies $q = upu^*$. Let $t \mapsto u(t)$ be a continuous path in $\mathcal{U}(\widetilde{\mathcal{C}})$ with $u(0) = \mathbf{1}$ and $u(1) = u$. Because $\mathcal{C}$ is an ideal in $\widetilde{\mathcal{C}}$ it follows that $u(t)pu(t)^*$ is a projection in $\mathcal{C}$ for any $t$, and thus the map $t \mapsto u(t)pu(t)^*$ is a continuous path in $\mathcal{P}(\mathcal{C})$ from $p$ to $q$.

Conversely, if $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$, then there are projections $p_0, p_1, \dots p_n$ in $\mathcal{C}$ with $p_0 = p$ and $p_n = q$ such that $\|p_{j+1} - p_j\| < 1/2$ for any $j = 0, 1, \dots n-1$. By concatenation, it is sufficient to show the statement for $\|p-q\| < 1/2$. Thus, let us set

$$b := pq + (\mathbf{1} - p)(\mathbf{1} - q) \in \widetilde{\mathcal{C}}$$

and observe that

$$pb = pq = bq, \tag{2.4}$$

and that

$$\|b - \mathbf{1}\| = \big\|p(q-p) + (\mathbf{1}-p)(p-q)\big\| \leq 2\|p-q\| < 1.$$

By Lemma 2.1.9 it follows that $b$ is invertible and that $b \sim_h \mathbf{1}$ in $\mathcal{GL}(\widetilde{\mathcal{C}})$. In addition, by considering the polar decomposition $b = u|b|$ with $u \in \mathcal{U}(\widetilde{\mathcal{C}})$, one obtains from (2.4) and from Proposition 2.2.7 that $p = uqu^*$. Finally, from Proposition 2.1.8.$(ii)$ one deduces that $u \sim_h b \sim_h \mathbf{1}$ in $\mathcal{GL}(\widetilde{\mathcal{C}})$, from which one gets that $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$, again from Proposition 2.1.8.$(iii)$. □

Up to now, we have considered three equivalence relation, the homotopy relation $\sim_h$, the Murray-von Neumann relation $\sim$ and the unitary relation $\sim_u$. It can be shown on examples that these three relations are different from each other, see for example [RLL00, Ex. 2.2.9]. In fact, in the next lemma we shall show that homotopy equivalence is stronger than unitary equivalence, which is itself stronger that Murray-von Neumann equivalence. However, we shall see subsequently that these relations are equal modulo passing to matrix algebras.

**Lemma 2.2.9.** *Let $p, q$ be projections in a $C^*$-algebra $\mathcal{C}$. Then:*

*(i) If $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$, then $p \sim_u q$,*

*(ii) If $p \sim_u q$, then $p \sim q$.*

*Proof.* Clearly, the first statement is a consequence of Proposition 2.2.8. For the second one, let $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$. Then $v := up$ belongs to $\mathcal{C}$ and satisfies $v^*v = p$ and $vv^* = upu^* = q$. □

**Proposition 2.2.10.** *Let $p, q$ be projections in a $C^*$-algebra $\mathcal{C}$. Then:*

*(i) If $p \sim q$, then $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_u \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_2(\mathcal{C})$,*

*(ii) If $p \sim_u q$, then $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathcal{P}\big(M_2(\mathcal{C})\big)$.*

Let us mention that both algebras $M_2(\widetilde{\mathcal{C}})$ and $\widetilde{M_2(\mathcal{C})}$ (the smallest unitization of $M_2(\mathcal{C})$) will be used during the proof of this proposition. It is easily observed that these two algebras are not equal, as illustrated in the following proof.

*Proof.* i) Let $v \in \mathcal{C}$ such that $p = v^*v$ and $q = vv^*$. By taking (2.2) into account and by denoting by $\mathbf{1}$ the unit of $\widetilde{\mathcal{C}}$, one readily infers that

$$u := \begin{pmatrix} v & \mathbf{1}-q \\ \mathbf{1}-p & v^* \end{pmatrix}, \qquad w := \begin{pmatrix} q & \mathbf{1}-q \\ \mathbf{1}-q & q \end{pmatrix}$$

are unitary elements of $M_2(\widetilde{\mathcal{C}})$, with $u^* = \left(\begin{smallmatrix} v^* & \mathbf{1}-p \\ \mathbf{1}-q & v \end{smallmatrix}\right)$ and $w^* = w$. Then, one observes that

$$wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* w^* = w \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, one has $wu \in M_2(\widetilde{\mathcal{C}})$, but by an explicit computation one observes that

$$wu = \begin{pmatrix} v + (\mathbf{1}-q)(\mathbf{1}-p) & (\mathbf{1}-q)v^* \\ q(\mathbf{1}-p) & (\mathbf{1}-q) + qv^* \end{pmatrix}$$

belongs to $\widetilde{M_2(\mathcal{C})} \subset M_2(\widetilde{\mathcal{C}})$, the claim $(i)$ is thus proved.

ii) Let $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$. By (2.1), there exists a continuous path $v : [0,1] \to \mathcal{U}\big(M_2(\widetilde{\mathcal{C}})\big)$ such that $v(0) = \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ and $v(1) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$. Then by setting $w(t) := v(t) \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) v(t)^*$, one gets that $w(t) \in \mathcal{P}\big(M_2(\mathcal{C})\big)$ for any $t$, that the map $t \mapsto w(t)$ is continuous, and that $w(0) = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $w(1) = \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$. □

## 2.3 Liftings

Let us now consider two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism. Given an element $b \in \mathcal{Q}$, an element $a \in \mathcal{C}$ satisfying $\varphi(a) = b$ is called *a lift for* $b$. The set of all lifts for $b$ is then given by $a + \mathsf{Ker}(\varphi)$. Now, if $b$ has some additional properties, like being a projection or a unitary element, we shall be interested in looking at lifts for $b$ which share similar properties (if possible). In the following statement, we collect several results in this direction.

**Proposition 2.3.1.** *Let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism between $C^*$-algebras. Then:*

*(i) Every $b \in \mathcal{Q}$ has a lift $a \in \mathcal{C}$ satisfying $\|a\| = \|b\|$,*

*(ii) Every self-adjoint $b \in \mathcal{Q}$ has a self-adjoint lift $a \in \mathcal{C}$. Moreover, this self-adjoint lift can be chosen such that $\|a\| = \|b\|$,*

*(iii) Every positive $b \in \mathcal{Q}$ has a positive lift $a \in \mathcal{C}$, and this lift can be chosen such that $\|a\| = \|b\|$,*

*(iv) A normal element $b \in \mathcal{Q}$ does not in general lift to a normal element in $\mathcal{C}$,*

*(v) A projection in $\mathcal{Q}$ does not in general lift to a projection in $\mathcal{Q}$,*

*(vi) When $\mathcal{C}$ and $\mathcal{Q}$ are unital, a unitary element $b \in \mathcal{Q}$ does not in general lift to a unitary element in $\mathcal{C}$.*

*Proof.* ii) Consider a self-adjoint element $b \in \mathcal{Q}$, and let $x \in \mathcal{C}$ be any lift for $b$. Then $a_0 := (x + x^*)/2$ defines a self-adjoint lift for $b$. In order to impose the equality of the norms, let us consider $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$f(t) = \begin{cases} -\|b\| & \text{if } t \leq -\|b\| \\ t & \text{if } -\|b\| \leq t \leq \|b\| \\ \|b\| & \text{if } t \geq \|b\| \end{cases}$$

and set $a := f(a_0)$. Then $a$ is self-adjoint, being obtained by functional calculus of a self-adjoint element, and one has $\sigma(a) = \{f(t) \mid t \in \sigma(a_0)\} \subset [-\|b\|, \|b\|]$. One infers from this inequality that $\|a\| \leq \|b\|$, since $r(a) = \|a\|$ for any self-adjoint element. On the other hand, one has

$$\varphi(a) = \varphi\big(f(a_0)\big) = f\big(\varphi(a_0)\big) = f(b) = b,$$

because of the definition of $f$. Since $\varphi$ is a $*$-homomorphism, one infers that $\|\varphi\| \leq 1$, from which one concludes that $\|b\| \leq \|a\|$. By collecting these inequalities one obtains that $\|a\| = \|b\|$.

i) Let $b$ be an arbitrary element of $\mathcal{Q}$, and set $y = \left(\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\right)$. Then $y$ is a self-adjoint element in $M_2(\mathcal{Q})$, and

$$\|y\|^2 = \|y^*y\| = \|\left(\begin{smallmatrix} bb^* & 0 \\ 0 & b^*b \end{smallmatrix}\right)\| = \max\{\|bb^*\|, \|b^*b\|\} = \|b\|^2.$$

It follows then by $(ii)$ that there exists a self-adjoint lift $x = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \in M_2(\mathcal{C})$ for $y$ with $\|x\| = \|y\| = \|b\|$. Clearly, $a := x_{12}$ is then a lift for $b$, and from (1.4) one infers that $\|a\| \leq \|x\| = \|b\|$. As in the proof of $(ii)$, one also has $\|b\| \leq \|a\|$, from which one deduces that $\|a\| = \|b\|$.

iii) Let $b$ be a positive element in $\mathcal{Q}$, and let $x \in \mathcal{C}$ be any lift for $b$. Set $a_0 := (x^*x)^{1/2}$, which is positive, and observe that

$$\varphi(a_0) = \big(\varphi(x)^*\varphi(x)\big)^{1/2} = (b^*b)^{1/2} = b.$$

We can then set $a := f(a_0)$ with the function $f$ introduced in the proof of $(ii)$, and one gets that $a$ is self-adjoint with $\sigma\big(f(a)\big) \subset [0, \|b\|]$. Thus, $a$ is positive and satisfies $\varphi(a) = b$ together with $\|a\| = \|b\|$.

The remaining three assertions are based on counterexamples. For $(iv)$, a counterexample is provided in [RLL00, Ex. 9.4.(iii)] and is based on the unilateral shift. For $(v)$, consider the algebras $\mathcal{C} := C\big([0,1]\big)$ and $\mathcal{Q} := \mathbb{C} \oplus \mathbb{C}$, with $\varphi : \mathcal{C} \to \mathcal{Q}$ defined by $\varphi(f) = \big(f(0), f(1)\big)$ for any $f \in \mathcal{C}$. Clearly, $(0,1)$ is a projection in $\mathcal{Q}$, but there is no lift $f$ in $\mathcal{C}$ which is a projection and which satisfies $\big(f(0), f(1)\big) = (0,1)$. For $(vi)$, a counterexample is provided in [RLL00, Ex. 2.12.(ii)] for the algebras $\mathcal{C} := C(\mathbb{D})$ and $\mathcal{Q} := C(\mathbb{T})$, with $\mathbb{D} := \{z \in \mathbb{C} \mid \|z\| \leq 1\}$. $\square$

# Chapter 3

# $K_0$-group for a unital $C^*$-algebra

In this chapter, we associate with each unital $C^*$-algebra an Abelian group. This group will be constructed from equivalence classes of projections. The $K_0$-group for non-unital $C^*$-algebra will be described in the next Chapter.

## 3.1 Semigroups of projections

Let us start by introducing a semigroup of projections in a $C^*$-algebra, with or without a unit. For that purpose, let $\mathcal{C}$ be an arbitrary $C^*$-algebra and set for $n \in \mathbb{N}^*$

$$\mathcal{P}_n(\mathcal{C}) := \mathcal{P}\big(M_n(\mathcal{C})\big) \qquad \text{and} \qquad \mathcal{P}_\infty(\mathcal{C}) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{C}).$$

One can then define the relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{C})$, namely for two elements $p, q \in \mathcal{P}_\infty(\mathcal{C})$ one writes $p \sim_0 q$ if there exits $v \in M_{m,n}(\mathcal{C})$ such that $p = v^*v \in \mathcal{P}_n(\mathcal{C})$ and $q = vv^* \in \mathcal{P}_m(\mathcal{C})$. Clearly, $M_{m,n}(\mathcal{C})$ denotes the set of $m \times n$ matrices with entries in $\mathcal{C}$, and the adjoint $v^*$ of $v \in M_{m,n}(\mathcal{C})$ is obtained by taking the transpose of the matrix, and then the adjoint of each entry.

One easily observes that the relation $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(\mathcal{C})$. It combines both the Murray-von Neumann equivalence relation $\sim$ and and the identification of projections in different sized matrix algebras over $\mathcal{C}$. For example, if $p, q \in \mathcal{P}_n(\mathcal{C})$ then $p \sim_0 q$ if and only if $p \sim q$.

We also define a binary operation $\oplus$ on $\mathcal{P}_\infty(\mathcal{C})$ by

$$p \oplus q = \operatorname{diag}(p, q) := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

so that $p \oplus q$ belongs to $\mathcal{P}_{m+n}(\mathcal{C})$ whenever $p \in \mathcal{P}_n(\mathcal{C})$ and $q \in \mathcal{P}_m(\mathcal{C})$. We can now derive some of the properties of $\sim_0$.

**Proposition 3.1.1.** *Let $\mathcal{C}$ be a $C^*$-algebra, and let $p, q, r, p', q'$ be elements of $\mathcal{P}_\infty(\mathcal{C})$. Then:*

*(i)* $p \sim_0 p \oplus 0_n$ *for any natural number* $n$*, where* $0_n$ *denotes the* 0*-element of* $M_n(\mathcal{C})$*,*

*(ii) If* $p \sim_0 p'$ *and* $q \sim_0 q'$*, then* $p \oplus q \sim_0 p' \oplus q'$*,*

*(iii)* $p \oplus q \sim_0 q \oplus p$*,*

*(iv) If* $p, q \in \mathcal{P}_n(\mathcal{C})$ *such that* $pq = 0$*, then* $p + q \in \mathcal{P}_n(\mathcal{C})$ *and* $p + q \sim_0 p \oplus q$*,*

*(v)* $(p \oplus q) \oplus r = p \oplus (q \oplus r)$*.*

*Proof.* i) Let $m, n$ be integers, and let $p \in \mathcal{P}_m(\mathcal{C})$. One then sets $v := \left(\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right) \in M_{m+n,m}(\mathcal{C})$, and one gets $p = v^*v$ and $vv^* = p \oplus 0_n$.

ii) Let $v, w$ such that $p = v^*v$, $p' = vv^*$, $q = w^*w$ and $q' = ww^*$, and set $u := \mathrm{diag}(v, w)$. Then $p \oplus q = u^*u$ and $p' \oplus q' = uu^*$.

iii) Assume $p \in \mathcal{P}_n(\mathcal{C})$ and $q \in \mathcal{P}_m(\mathcal{C})$, and set $v := \left(\begin{smallmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{smallmatrix}\right)$, with $0_{k,l}$ the 0-matrix of size $k \times l$. Then one gets $p \oplus q = v^*v$ and $q \oplus p = vv^*$.

iv) If $pq = 0$ it is easily observe that $p + q$ is itself a projection. Then, if one sets $v := \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in M_{2n,n}(\mathcal{C})$, one gets $p + q = v^*v$ and $p \oplus q = vv^*$.

v) This last statement is trivial. □

**Definition 3.1.2.** *For any* $C^*$*-algebra* $\mathcal{C}$*, one sets*

$$\mathcal{D}(\mathcal{C}) := \mathcal{P}_\infty(\mathcal{C}) / \sim_0$$

*which corresponds to the equivalent classes of elements of* $\mathcal{P}_\infty(\mathcal{C})$ *modulo the equivalence relation* $\sim_0$*. For any* $p \in \mathcal{P}_\infty(\mathcal{C})$ *one writes* $[p]_\mathcal{D} \in \mathcal{D}(\mathcal{C})$ *for the equivalent class containing* $p$*. The set* $\mathcal{D}(\mathcal{C})$ *is endowed with a binary operation defined for any* $p, q \in \mathcal{P}_\infty(\mathcal{C})$ *by*

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}. \tag{3.1}$$

Because of the previous proposition, one directly infers the following result:

**Lemma 3.1.3.** *The pair* $\big(\mathcal{D}(\mathcal{C}), +\big)$ *defines an Abelian semigroup.*

We end this section with two exercises dealing with projections.

**Exercise 3.1.4.** *Let* $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ *denote the usual trace on square matrices, and let* $p, q \in \mathcal{P}\big(M_n(\mathbb{C})\big)$*. Show that the following statements are equivalent:*

*(i)* $p \sim q$*,*

*(ii)* $\mathrm{tr}(p) = \mathrm{tr}(q)$*,*

*(iii)* $\dim\big(p(\mathbb{C}^n)\big) = \dim\big(q(\mathbb{C}^n)\big)$*.*

*Use this to show that* $\mathcal{D}(\mathbb{C}) \cong \mathbb{Z}_+ \equiv \{0, 1, 2, \dots\}$ *when* $\mathbb{Z}_+$ *is equipped with the usual addition.*

**Exercise 3.1.5.** *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let $p, q$ be projections in $\mathcal{B}(\mathcal{H})$.*

*(i) Show that $p \sim q$ if and only if $\dim\big(p(\mathcal{H})\big) = \dim\big(q(\mathcal{H})\big)$,*

*(ii) Show that $p \sim_u q$ if and only if $\dim\big(p(\mathcal{H})\big) = \dim\big(q(\mathcal{H})\big)$ and $\dim\big(p(\mathcal{H})^\perp\big) = \dim\big(q(\mathcal{H})^\perp\big)$,*

*(iii) Infer that $\mathcal{D}\big(\mathcal{B}(\mathcal{H})\big) \cong \mathbb{Z}_+ \cup \{\infty\} \equiv \{0, 1, 2, \ldots, \infty\}$, where the usual addition on $\mathbb{Z}_+$ is considered together with the addition $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{Z}_+ \cup \{\infty\}$.*

## 3.2 The $K_0$-group

In this section we construct the $K_0$-group associated with a unital $C^*$-algebra $\mathcal{C}$. This group is defined in terms of the Grothendieck construction applied to the Abelian semigroup $\big(\mathcal{D}(\mathcal{C}), +\big)$. We first recall this construction in an abstract setting.

Let $(\mathcal{D}, +)$ be an Abelian semigroup, and define on $\mathcal{D}\times\mathcal{D}$ the relation $\sim$ by $(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in \mathcal{D}$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. This relation is clearly reflexive and symmetric. For the transitivity, suppose that $(x_1, y_1) \sim (x_2, y_2)$ and that $(x_2, y_2) \sim (x_3, y_3)$. This means that there exist $z, w \in \mathcal{D}$ such that

$$x_1 + y_2 + z = x_2 + y_1 + z \qquad \text{and} \qquad x_2 + y_3 + w = x_3 + y_2 + w.$$

It then follows that

$$x_1 + y_3 + (y_2 + z + w) = x_2 + y_1 + z + y_3 + w = x_3 + y_1 + (y_2 + z + w)$$

so that $(x_1, y_1) \sim (x_3, y_3)$. As a consequence, $\sim$ defines an equivalence relation on $\mathcal{D}\times\mathcal{D}$. The equivalence class containing $(x, y)$ is denoted by $\langle x, y\rangle$, and we set $\mathcal{G}(\mathcal{D})$ for the quotient $\mathcal{D} \times \mathcal{D}/\sim$. Then, the operation

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$$

endows $\mathcal{G}(\mathcal{D})$ with the structure of an Abelian group. Indeed, the inverse $-\langle x, y\rangle$ of $\langle x, y\rangle$ is given by $\langle y, x\rangle$, and $\langle x, x\rangle = 0$, for any $x, y \in \mathcal{D}$. The pair $\big(\mathcal{G}(\mathcal{D}), +\big)$ is called *the Grothendieck group.*

For any fixed $y \in \mathcal{D}$, let us also define the map

$$\gamma_{\mathcal{D}} : \mathcal{D} \ni x \mapsto \gamma_{\mathcal{D}}(x) := \langle x + y, y\rangle \in \mathcal{G}(\mathcal{D}),$$

and observe that this map does not depend on the choice of any specific $y \in \mathcal{D}$. Indeed, one easily observes that $(x+y, y)$ and $(x+y', y')$ define the same equivalence class since $(x + y) + y' = (x + y') + y$. The map $\gamma_{\mathcal{D}}$ is called *the Grothendieck map.*

Finally, one says that the semigroup $(\mathcal{D}, +)$ has the *cancellation property* if whenever the equality $x+z = y+z$ holds, it follows that $x = y$. Let us now gather some additional information on this construction in the following proposition.

**Proposition 3.2.1.** *Let $(\mathcal{D},+)$ be an Abelian semigroup, and let $\big(\mathcal{G}(\mathcal{D}),+\big)$ and $\gamma_{\mathcal{D}}$ be the corresponding Grothendieck group and Grothendieck map. Then:*

*(i)* Universal property: *If $H$ is an Abelian group and if $\varphi : \mathcal{D} \to H$ is an additive map, then there is one and only one group homomorphism $\psi : \mathcal{G}(\mathcal{D}) \to H$ making the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\varphi"] \arrow[dr, "\gamma_{\mathcal{D}}"'] & H \\ & \mathcal{G}(\mathcal{D}) \arrow[u, "\psi"'] \end{tikzcd}$$

*commutative,*

*(ii)* Functoriality: *For every additive map $\varphi : \mathcal{D} \to \mathcal{D}'$ between semigroups there exists one and only one group morphism $\mathcal{G}(\varphi) : \mathcal{G}(\mathcal{D}) \to \mathcal{G}(\mathcal{D}')$ making the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\varphi"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{D}' \arrow[d, "\gamma_{\mathcal{D}'}"] \\ \mathcal{G}(\mathcal{D}) \arrow[r, "\mathcal{G}(\varphi)"'] & \mathcal{G}(\mathcal{D}') \end{tikzcd}$$

*commutative,*

*(iii)* $\mathcal{G}(\mathcal{D}) = \{\gamma_{\mathcal{D}}(x) - \gamma_{\mathcal{D}}(y) \mid x, y \in \mathcal{D}\}$,

*(iv)* *For any $x, y \in \mathcal{D}$ one has $\gamma_{\mathcal{D}}(x) = \gamma_{\mathcal{D}}(y)$ if and only if $x + z = y + z$ for some $z \in \mathcal{D}$,*

*(v)* *The Grothendieck map $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{G}(\mathcal{D})$ is injective if and only if $(\mathcal{D},+)$ has the cancellation property,*

*(vi)* *Let $(H,+)$ be an Abelian group, and let $\mathcal{D}$ be a non-empty subset of $H$. If $\mathcal{D}$ is closed under addition, then $(\mathcal{D},+)$ is an Abelian semigroup with the cancellation property. In addition, $\mathcal{G}(\mathcal{D})$ is isomorphic to the subgroup $H_0$ generated by $\mathcal{D}$, and $H_0 = \{x - y \mid x, y \in \mathcal{D}\}$.*

The proofs of these statements can be found in [RLL00, Sec. 3.1.2]. Let us just mention the one of *(iii)*: Since each element of $\mathcal{G}(\mathcal{D})$ has the form $\langle x, y\rangle$ for some $x, y \in \mathcal{D}$, it is sufficient to observe that

$$\langle x, y\rangle = \langle x + y, y\rangle - \langle x + y, x\rangle = \gamma_{\mathcal{D}}(x) - \gamma_{\mathcal{D}}(y).$$

We still illustrate the previous construction with two examples.

**Examples 3.2.2.** *(i) The Grothendieck group of the Abelian semigroup $(\mathbb{Z}_+,+)$ is isomorphic to $(\mathbb{Z},+)$. Note that $(\mathbb{Z}_+,+)$ has the cancellation property.*

*(ii) The Grothendieck group of the Abelian semigroup* $\big(\mathbb{Z}_+ \cup \{\infty\}, +\big)$ *is* $\{0\}$*. Note that* $\big(\mathbb{Z}_+ \cup \{\infty\}, +\big)$ *does not possess the cancellation property.*

We are now ready for the main definition of this chapter. Recall that for any $C^*$-algebra $\mathcal{C}$, the Abelian semigroup $\big(\mathcal{D}(\mathcal{C}), +\big)$ has been introduced in Definition 3.1.2, see also Lemma 3.1.3.

**Definition 3.2.3.** *Let* $\mathcal{C}$ *be a unital* $C^*$*-algebra, and let* $\big(\mathcal{D}(\mathcal{C}), +\big)$ *be the corresponding Abelian semigroup. The Abelian group* $K_0(\mathcal{C})$ *is defined by*

$$K_0(\mathcal{C}) := \mathcal{G}\big(\mathcal{D}(\mathcal{C})\big).$$

*One also set* $[\cdot]_0 : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{C})$ *for any* $p \in \mathcal{P}_\infty(\mathcal{C})$ *by*

$$[p]_0 := \gamma\big([p]_{\mathcal{D}}\big)$$

*with* $\gamma : \mathcal{D}(\mathcal{C}) \to K_0(\mathcal{C})$ *the Grothendieck map.*

In the following two propositions, we provide a standard picture of the $K_0$-group for a unital $C^*$-algebra, and state some of its universal properties. Before them, we introduce one more equivalence relation on $\mathcal{P}_\infty(\mathcal{C})$, namely $p, q \in \mathcal{P}_\infty(\mathcal{C})$ are *stable equivalent*, written $p \sim_s q$, if there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $p \oplus r \sim_0 q \oplus r$. Note that if $\mathcal{C}$ is unital, then $p \sim_s q$ if and only if $p \oplus \mathbf{1}_n \sim_0 q \oplus \mathbf{1}_n$ for some $n \in \mathbb{N}$. Indeed, if $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathcal{P}_n(\mathcal{C})$, then

$$p \oplus \mathbf{1}_n \sim_0 p \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus r \oplus (\mathbf{1}_n - r) \sim_0 q \oplus \mathbf{1}_n,$$

where Proposition 3.1.1.$(iv)$ has been used twice.

**Proposition 3.2.4.** *For any unital* $C^*$*-algebra* $\mathcal{C}$ *one has*

$$K_0(\mathcal{C}) = \big\{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(\mathcal{C})\big\} = \big\{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(\mathcal{C}), n \in \mathbb{N}^*\big\}. \tag{3.2}$$

*Moreover, one has*

*(i)* $[p \oplus q]_0 = [p]_0 + [q]_0$ *for any projections* $p, q \in \mathcal{P}_\infty(\mathcal{C})$*,*

*(ii)* $[0_{\mathcal{C}}]_0 = 0$*, where* $0_{\mathcal{C}}$ *stands for the zero element of* $\mathcal{C}$*,*

*(iii) If* $p, q \in \mathcal{P}_n(\mathcal{C})$ *for some* $n \in \mathbb{N}^*$ *and if* $p \sim_h q \in \mathcal{P}_n(\mathcal{C})$*, then* $[p]_0 = [q]_0$*,*

*(iv) If* $p, q$ *are mutually orthogonal projections in* $\mathcal{P}_n(\mathcal{C})$*, then* $[p + q]_0 = [p]_0 + [q]_0$*,*

*(v) For all* $p, q \in \mathcal{P}_\infty(\mathcal{C})$*, then* $[p]_0 = [q]_0$ *if and only if* $p \sim_s q$*.*

*Proof.* The first equality in (3.2) follows from Proposition 3.2.1.(*iii*). Hence, if $g$ is an element of $K_0(\mathcal{C})$ there exist $p' \in \mathcal{P}_k(\mathcal{C})$ and $q' \in \mathcal{P}_l(\mathcal{C})$ such that $g = [p']_0 - [q']_0$. Choose then $n$ greater than $k$ and $l$, and set $p = p' \oplus 0_{n-k}$ and $q := q \oplus 0_{n-l}$. Then $p, q \in \mathcal{P}_n(\mathcal{C})$ with $p \sim_0 p'$ and $q \sim_0 q'$ by Proposition 3.1.1.(*i*). It thus follows that $g = [p]_0 - [q]_0$.

i) One has by (3.1)

$$[p \oplus q]_0 = \gamma\big([p \oplus q]_{\mathcal{D}}\big) = \gamma\big([p]_{\mathcal{D}} + [q]_{\mathcal{D}}\big) = \gamma\big([p]_{\mathcal{D}}\big) + \gamma\big([q]_{\mathcal{D}}\big) = [p]_0 + [q]_0.$$

ii) Since $0_{\mathcal{C}} \oplus 0_{\mathcal{C}} \sim_0 0_{\mathcal{C}}$, point (*i*) yields that $[0_{\mathcal{C}}]_0 + [0_{\mathcal{C}}]_0 = [0_{\mathcal{C}}]_0$, which means that $[0_{\mathcal{C}}]_0 = 0$.

iii) This statement follows from the implications

$$p \sim_h q \Rightarrow p \sim q \Rightarrow p \sim_0 q \Leftrightarrow [p]_{\mathcal{D}} = [q]_{\mathcal{D}} \Rightarrow [p]_0 = [q]_0,$$

where the first two relations are defined only when $p$ and $q$ are in the same matrix algebra over $\mathcal{C}$, while the three other implications hold for any $p, q \in \mathcal{P}_\infty(\mathcal{C})$. Note that the first implication is due to Lemma 2.2.9.

iv) By Proposition 3.1.1.(*iv*), one has $p + q \sim_0 p \oplus q$, and therefore $[p + q]_0 = [p \oplus q]_0 = [p]_0 + [q]_0$ by (*i*).

v) If $[p]_0 = [q]_0$, then by Proposition 3.2.1.(*iv*) there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}}$. Hence $[p \oplus r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}}$, and then $p \oplus r \sim_0 q \oplus r$. It thus follows that $p \sim_s q$.

Conversely, if $p \sim_s q$, then there exists $r \in \mathcal{P}_\infty(\mathcal{C})$ such that $p \oplus r \sim_0 q \oplus r$. By (*i*) one gets that $[p]_0 + [r]_0 = [q]_0 + [r]_0$, and because $K_0(\mathcal{C})$ is a group we conclude that $[p]_0 = [q]_0$. □

**Proposition 3.2.5** (Universal property of $K_0$)**.** *Let $\mathcal{C}$ be a unital $C^*$-algebra, and let $H$ be an Abelian group. Suppose that there exists $\nu : \mathcal{P}_\infty(\mathcal{C}) \to H$ satisfying the three conditions:*

*(i)* $\nu(p \oplus q) = \nu(p) + \nu(q)$ *for any* $p, q \in \mathcal{P}_\infty(\mathcal{C})$,

*(ii)* $\nu(0_{\mathcal{C}}) = 0$,

*(iii) If* $p, q \in \mathcal{P}_n(\mathcal{C})$ *for some* $n \in \mathbb{N}^*$ *and if* $p \sim_h q \in \mathcal{P}_n(\mathcal{C})$, *then* $\nu(p) = \nu(q)$.

*Then there exists a unique group homomorphism $\alpha : K_0(\mathcal{C}) \to H$ such that the diagram*

$$\begin{tikzcd} \mathcal{P}_\infty(\mathcal{C}) \arrow[d, "{[\cdot]_0}"'] \arrow[dr, "\nu"] & \\ K_0(\mathcal{C}) \arrow[r, "\alpha"'] & H \end{tikzcd}$$

*is commutative.*

The proof of this statement is provided the proof of [RLL00, Prop. 3.1.8] to which we refer.

## 3.3 Functoriality of $K_0$

Let us now consider two unital $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a $*$-homomorphism. As already seen in Section 1.3, $\varphi$ extends to a $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ for any $n \in \mathbb{N}^*$. Again, the same notation is used for the original morphism and for its extensions. Since $*$-homomorphisms map projections to projections, one infers that $\varphi$ maps $\mathcal{P}_\infty(\mathcal{C})$ into $\mathcal{P}_\infty(\mathcal{Q})$. Let us then define the map $\nu : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{Q})$ by $\nu(p) := [\varphi(p)]_0$ for any $p \in \mathcal{P}_\infty(\mathcal{C})$. Since $\nu$ satisfies the three conditions of Proposition 3.2.5 with $H = K_0(\mathcal{Q})$ one infers that there exists a unique group homomorphism $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ given by

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0 \tag{3.3}$$

for any $p \in \mathcal{P}_\infty(\mathcal{C})$. In other words, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{P}_\infty(\mathcal{C}) \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_0}"'] & \mathcal{P}_\infty(\mathcal{Q}) \arrow[d, "{[\cdot]_0}"] \\
K_0(\mathcal{C}) \arrow[r, "K_0(\varphi)"'] & K_0(\mathcal{Q}).
\end{tikzcd}$$

With this construction at hand, we can now state and prove the main result on functoriality. Here, the functor $K_0$ associates with any unital $C^*$-algebra $\mathcal{C}$ the Abelian group $K_0(\mathcal{C})$. For two unital $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$ one sets $0_{\mathcal{C}\to\mathcal{Q}}$ for the map sending all elements of $\mathcal{C}$ to $0 \in \mathcal{Q}$, and $0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$ for the map sending all elements of $K_0(\mathcal{C})$ to the identity element in $K_0(\mathcal{Q})$.

**Proposition 3.3.1** (Functoriality of $K_0$ (unital case))**.** *Let $\mathcal{J}$, $\mathcal{C}$ and $\mathcal{Q}$ be unital $C^*$-algebras. Then*

*(i)* $K_0(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{K_0(\mathcal{C})}$,

*(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are $*$-homomorphisms, then*

$$K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi),$$

*(iii)* $K_0(\{0\}) = \{0\}$,

*(iv)* $K_0(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$.

*Proof.* By using (3.3) one can check that for any $p \in \mathcal{P}_\infty(\mathcal{C})$ and any $q \in \mathcal{P}_\infty(\mathcal{J})$ the equalities

$$K_0(\mathrm{id}_{\mathcal{C}})([p]_0) = [p]_0, \qquad K_0(\psi \circ \varphi)([q]_0) = \big(K_0(\psi) \circ K_0(\varphi)\big)([q]_0)$$

hold. Then, by taking the standard picture of $K_0$ (equality (3.2)) into account, one readily deduces the statement $(i)$ and $(ii)$.

iii) One has $\mathcal{P}_n(\{0\}) = \{0_n\}$, with $0_n$ the zero (and single) element of $M_n(\{0\})$. Since the zero projections $0 = 0_1, 0_2, \ldots$ are all $\sim_0$-equivalent, it follows that $\mathcal{D}(\{0\}) = \{[0]_{\mathcal{D}}\}$. As a consequence, one deduces that $K_0(\{0\}) = \mathcal{G}\big(\{[0]_{\mathcal{D}}\}\big) = \{0\}$.

iv) Since $0_{\mathcal{C}\to\mathcal{Q}} = 0_{0\to\mathcal{Q}} \circ 0_{\mathcal{C}\to 0} : \mathcal{C} \to \{0\} \to \mathcal{Q}$, the statement $(iv)$ can be deduced from $(ii)$ and $(iii)$. □

For two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, two $*$-homomorphisms $\varphi_0 : \mathcal{C} \to \mathcal{Q}$ and $\varphi_1 : \mathcal{C} \to \mathcal{Q}$ are said to be *homotopic*, written $\varphi_0 \sim_h \varphi_1$, if there exists a path of $*$-homomorphisms $t \mapsto \varphi(t)$ with $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$ such that for any $a \in \mathcal{C}$ the map $[0,1] \ni t \mapsto [\varphi(t)](a) \in \mathcal{Q}$ is continuous. In this case, one also says that $t \mapsto \varphi(t)$ is *pointwise continuous*. The two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$ are said to be *homotopy equivalent* if there exist two $*$-homomorphisms $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{Q} \to \mathcal{C}$ such that $\psi \circ \varphi \sim_h \mathrm{id}_{\mathcal{C}}$ and $\varphi \circ \psi \sim_h \mathrm{id}_{\mathcal{Q}}$. In this case one says that

$$\mathcal{C} \xrightarrow{\varphi} \mathcal{Q} \xrightarrow{\psi} \mathcal{C} \tag{3.4}$$

is a *homotopy* between $\mathcal{C}$ and $\mathcal{Q}$.

**Proposition 3.3.2** (Homotopy invariance of $K_0$ (unital case))**.** *Let $\mathcal{C}$ and $\mathcal{Q}$ be unital $C^*$-algebras.*

*(i) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homomorphisms, then $K_0(\varphi) = K_0(\psi)$,*

*(ii) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent, then $K_0(\mathcal{C})$ is isomorphic to $K_0(\mathcal{Q})$. More specifically, if (3.4) is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ and $K_0(\psi) : K_0(\mathcal{Q}) \to K_0(\mathcal{C})$ are isomorphisms, with $K_0(\varphi)^{-1} = K_0(\psi)$.*

**Exercise 3.3.3.** *Provide a proof of Proposition 3.3.2, with the possible help of [RLL00, Prop. 3.2.6].*

Our next aim is to show that $K_0$ preserves exactness of the short exact sequence obtained by adjoining a unit to a unital $C^*$-algebra. This result will be useful when defining the $K_0$-group for a non-unital $C^*$-algebra.

For two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, two $*$-homomorphisms $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are said to be *orthogonal to each other* or *mutually orthogonal*, written $\varphi \perp \psi$, if $\varphi(a)\psi(b) = 0$ for any $a, b \in \mathcal{C}$.

**Lemma 3.3.4.** *If $\mathcal{C}$ and $\mathcal{Q}$ are unital $C^*$-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are mutually orthogonal $*$-homomorphisms, then $\varphi + \psi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, and $K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$.*

*Proof.* One readily check that $\varphi + \psi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism. In addition, the $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ and $\psi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ are also orthogonal, for any $n \in \mathbb{N}^*$. By using then Proposition 3.2.4.$(iv)$ we obtain for any $p \in \mathcal{P}_n(\mathcal{C})$:

$$\begin{aligned} K_0(\varphi + \psi)([p]_0) &= [(\varphi + \psi)(p)]_0 = [\varphi(p) + \psi(p)]_0 \\ &= [\varphi(p)]_0 + [\psi(p)]_0 = K_0(\varphi)([p]_0) + K_0(\psi)([p]_0). \end{aligned}$$

This shows that $K_0(\varphi+\psi) = K_0(\varphi) + K_0(\psi)$. $\square$

**Lemma 3.3.5.** *For any unital $C^*$-algebra $\mathcal{C}$, the split exact sequence*

$$0 \longrightarrow \mathcal{C} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

*induces a split exact sequence*

$$0 \longrightarrow K_0(\mathcal{C}) \xrightarrow{K_0(\iota)} K_0(\widetilde{\mathcal{C}}) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow 0 \tag{3.5}$$

*Proof.* Recall from the proof of Lemma 2.2.4 that if $\tilde{\mathbf{1}}$ denotes the unit of $\widetilde{\mathcal{C}}$ and if $\mathbf{1}$ denotes the unit of $\mathcal{C}$, then $1 := \tilde{\mathbf{1}} - \mathbf{1}$ is a projection in $\widetilde{\mathcal{C}}$. In addition, $\widetilde{\mathcal{C}} = \mathcal{C} + \mathbb{C}1$, with $a1 = 1a = 0$ for any $a \in \mathcal{C}$. Let us then define the $*$-homomorphisms $\mu : \widetilde{\mathcal{C}} \to \mathcal{C}$ and $\lambda' : \mathbb{C} \to \widetilde{\mathcal{C}}$ by $\mu(a+\alpha 1) := a$ and $\lambda'(\alpha) := \alpha 1$ for any $a \in \mathcal{C}$ and $\alpha \in \mathbb{C}$. One readily infers that

$$\mathrm{id}_{\mathcal{C}} = \mu \circ \iota, \quad \mathrm{id}_{\widetilde{\mathcal{C}}} = \iota \circ \mu + \lambda' \circ \pi, \quad \pi \circ \iota = 0_{\mathcal{C}\to\mathbb{C}}, \quad \pi \circ \lambda = \mathrm{id}_{\mathbb{C}},$$

and the $*$-homomorphisms $\iota \circ \mu$ and $\lambda' \circ \pi$ are orthogonal to each other. Proposition 3.3.1 and Lemma 3.3.4 then lead to

$$\begin{aligned}
0_{K_0(\mathcal{C})\to K_0(\mathbb{C})} &= K_0(0_{\mathcal{C}\to\mathbb{C}}) = K_0(\pi) \circ K_0(\iota), && (3.6)\\
\mathrm{id}_{K_0(\mathbb{C})} &= K_0(\mathrm{id}_{\mathbb{C}}) = K_0(\pi \circ \lambda) = K_0(\pi) \circ K_0(\lambda), && (3.7)\\
\mathrm{id}_{K_0(\mathcal{C})} &= K_0(\mathrm{id}_{\mathcal{C}}) = K_0(\mu \circ \iota) = K_0(\mu) \circ K_0(\iota), && (3.8)\\
\mathrm{id}_{K_0(\widetilde{\mathcal{C}})} &= K_0(\mathrm{id}_{\widetilde{\mathcal{C}}}) = K_0(\iota \circ \mu + \lambda' \circ \pi) \\
&= K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi). && (3.9)
\end{aligned}$$

Now, the split exactness of (3.5) follows from these equalities. Indeed, the injectivity of $K_0(\iota)$ follows from (3.8). If $g \in \mathsf{Ker}\big(K_0(\pi)\big)$, one infers from (3.9) that $g = K_0(\iota)\big(K_0(\mu)(g)\big)$, which shows that $g$ belongs to $\mathsf{Ran}\big(K_0(\iota)\big)$. Since by (3.6) one also gets $\mathsf{Ran}\big(K_0(\iota)\big) \subset \mathsf{Ker}\big(K_0(\pi)\big)$, these two inclusions mean that $\mathsf{Ran}\big(K_0(\iota)\big) = \mathsf{Ker}\big(K_0(\pi)\big)$. Finally, the surjectivity of $K_0(\pi)$ is a by-product of (3.7), from which one also infers the splitness. $\square$

## 3.4 Examples

In this section, we introduce the examples discussed in [RLL00, Sec. 3.3] and refer to this book for the proofs.

Consider first a $C^*$-algebra $\mathcal{C}$ endowed with *a bounded trace* $\tau$, *i.e.* $\tau : \mathcal{C} \to \mathbb{C}$ is a bounded linear map satisfying the *trace property*

$$\tau(ab) = \tau(ba), \qquad \forall a, b \in \mathcal{C}.$$

This trace property implies in particular that $\tau(p) = \tau(q)$ whenever $p, q$ are Murray-von Neumann equivalent projections in $\mathcal{C}$. This trace is also called *positive* if $\tau(a) \geq 0$ whenever $a \in \mathcal{C}^+$. If $\mathcal{C}$ is unital and if $\tau(\mathbf{1}_{\mathcal{C}}) = 1$, then $\tau$ is called a *tracial state*.

For any trace $\tau$ on a $C^*$-algebra $\mathcal{C}$, one defines a trace on $M_n(\mathcal{C})$ by setting

$$\tau \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} \tau(a_{jj}).$$

It thus endows $\mathcal{P}_\infty(\mathcal{C})$ with a map $\tau : \mathcal{P}_\infty(\mathcal{C}) \to \mathbb{C}$, and this map satisfies the three conditions of Proposition 3.2.5. For the last one, recall that the homotopy equivalence implies the Murray-von Neumann equivalence, see Lemma 2.2.9. As a consequence, one infers that there exists a unique group homomorphism $K_0(\tau) : K_0(\mathcal{C}) \to \mathbb{C}$ satisfying for any $p \in \mathcal{P}_\infty(\mathcal{C})$

$$K_0(\tau)([p]_0) = \tau(p). \tag{3.10}$$

Note that if $\tau$ is positive, then the r.h.s. of (3.10) is a positive real number, and $K_0(\tau)$ maps $K_0(\mathcal{C})$ into $\mathbb{R}$.

**Example 3.4.1.** *For any $n \in \mathbb{N}^*$, one has*

$$K_0\big(M_n(\mathbb{C})\big) \cong \mathbb{Z}. \tag{3.11}$$

*In fact, if* tr *denotes the usual trace already introduced in Exercise 3.1.4, then*

$$K_0(\mathrm{tr}) : K_0\big(M_n(\mathbb{C})\big) \to \mathbb{Z} \tag{3.12}$$

*is an isomorphism.*

**Example 3.4.2.** *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then we have*

$$K_0\big(\mathcal{B}(\mathcal{H})\big) = \{0\}.$$

*Note that this fact is related to the content of Exercise 3.1.5.*

**Example 3.4.3.** *If $\Omega$ is a compact, connected and Hausdorff space, then there exists a surjective group homomorphism*

$$\dim : K_0\big(C(\Omega)\big) \to \mathbb{Z} \tag{3.13}$$

*which satisfies for $p \in \mathcal{P}_\infty\big(C(\Omega)\big)$ and $x \in \Omega$*

$$\dim([p]_0) = \mathrm{tr}\big(p(x)\big).$$

*Note that by continuity this number is independent of $x$. Note also that if $\Omega$ is* contractible[1] *then the map* (3.13) *is an isomorphism.*

**Exercise 3.4.4.** *Provide the proofs for the statements of Examples 3.4.1, 3.4.2 and 3.4.3.*

**Extension 3.4.5.** *Study the $K$-theory for topological spaces, as presented for example in [RLL00, Sec. 3.3.7].*

[1]The space $\Omega$ is contractible if there exists $x_0 \in \Omega$ and a continuous map $\alpha : [0,1] \times \Omega \to \Omega$ such that $\alpha(1, x) = x$ and $\alpha(0, x) = x_0$ for any $x \in \Omega$.

# Chapter 4

# $K_0$-group for an arbitrary $C^*$-algebra

In this chapter, we extend the construction of the $K_0$-group for a non-unital $C^*$-algebra, and show that this definition is coherent with the previous one when the algebra has a unit.

## 4.1 Definition and functoriality of $K_0$

**Definition 4.1.1.** *Let $\mathcal{C}$ be a non-unital $C^*$-algebra, and consider the associated split exact sequence*

$$0 \longrightarrow \mathcal{C} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

*One defines $K_0(\mathcal{C})$ as the kernel of the homomorphism $K_0(\pi) : K_0(\widetilde{\mathcal{C}}) \to K_0(\mathbb{C})$.*

Clearly, $K_0(\mathcal{C})$ is an Abelian group, being a subgroup of the Abelian group $K_0(\widetilde{\mathcal{C}})$. In addition, consider $p \in \mathcal{P}_\infty(\mathcal{C})$ and the equivalence class $[p]_0 \in K_0(\widetilde{\mathcal{C}})$. Since by (3.3) one has

$$K_0(\pi)([p]_0) = [\pi(p)]_0 = 0,$$

it follows that $[p]_0$ belongs to $K_0(\mathcal{C})$. In this way, we obtain a map $[\cdot]_0 : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{C})$.

Now, for any $C^*$-algebra, unital or not, we have a short exact sequence

$$0 \longrightarrow K_0(\mathcal{C}) \longrightarrow K_0(\widetilde{\mathcal{C}}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0. \tag{4.1}$$

Note that the map $K_0(\mathcal{C}) \longrightarrow K_0(\widetilde{\mathcal{C}})$ corresponds to $K_0(\iota)$ when $\mathcal{C}$ is unital while it simply corresponds to the inclusion map when $\mathcal{C}$ is not unital. Note also that in the unital case, it has been proved in Lemma 3.3.5 that (4.1) is indeed a short exact sequence while for the non-unital case, this follows from the definition of $K_0(\mathcal{C})$.

When $\mathcal{C}$ is unital, $K_0(\mathcal{C})$ is isomorphic to its image in $K_0(\widetilde{\mathcal{C}})$ through the map $K_0(\iota)$, and $K_0(\iota)$ maps $[p]_0 \in K_0(\mathcal{C})$ to $[p]_0 \in K_0(\widetilde{\mathcal{C}})$ for any $p \in \mathcal{P}_\infty(\mathcal{C})$. Since the image of

$K_0(\iota)$ is equal to the kernel of $K_0(\pi)$, the identity

$$K_0(\mathcal{C}) = \mathsf{Ker}\left(K_0(\pi)\right)$$

holds, for both unital and non-unital $C^*$-algebras (with a slight abuse of notation).

Let us now consider a $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between $C^*$-algebras, and let $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ be the corresponding $*$-homomorphism introduced right after Exercise 1.1.10. The commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{C}} \arrow[r, "\pi_{\mathcal{C}}"] \arrow[d, "\tilde{\varphi}"'] & \mathbb{C} \arrow[d, "\mathrm{id}"] \\
\mathcal{Q} \arrow[r, "\iota_{\mathcal{Q}}"'] & \widetilde{\mathcal{Q}} \arrow[r, "\pi_{\mathcal{Q}}"'] & \mathbb{C}.
\end{tikzcd}$$

induces by functoriality of $K_0$ for unital $C^*$-algebras the following commutative diagram:

$$\begin{tikzcd}
K_0(\mathcal{C}) \arrow[r] \arrow[d, "K_0(\varphi)"'] & K_0(\widetilde{\mathcal{C}}) \arrow[r, "K_0(\pi_{\mathcal{C}})"] \arrow[d, "K_0(\tilde{\varphi})"'] & K_0(\mathbb{C}) \arrow[d, "\mathrm{id}_{K_0(\mathbb{C})}"] \\
K_0(\mathcal{Q}) \arrow[r] & K_0(\widetilde{\mathcal{Q}}) \arrow[r, "K_0(\pi_{\mathcal{Q}})"'] & K_0(\mathbb{C})
\end{tikzcd}$$

where $K_0(\varphi)$ corresponds to the restriction to $K_0(\mathcal{C})$ of the group homomorphism $K_0(\tilde{\varphi}) : K_0(\widetilde{\mathcal{C}}) \to K_0(\widetilde{\mathcal{Q}})$. Note that if $\mathcal{C}$ and $\mathcal{Q}$ are unital, then the above group homomorphism $K_0(\varphi)$ corresponds to the one already introduced in Section 3.3. Note also that the equality

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0 \qquad \forall p \in \mathcal{P}_\infty(\mathcal{C})$$

holds, no matter if $\mathcal{C}$ is unital or not.

We can now state in a greater generality the functorial properties of $K_0$ which have already been discussed in Proposition 3.3.1 for unital $C^*$-algebras only. The proof of this statement consists in minor modifications of the one already presented in the unital case.

**Proposition 4.1.2** (Functoriality of $K_0$ (general case))**.** *Let $\mathcal{J}$, $\mathcal{C}$ and $\mathcal{Q}$ be $C^*$-algebras. Then*

*(i)* $K_0(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{K_0(\mathcal{C})}$,

*(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are $*$-homomorphisms, then*

$$K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi),$$

*(iii)* $K_0(\{0\}) = \{0\}$,

*(iv)* $K_0(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$.

Let us now mention that the homotopy invariance of $K_0$, as already presented in Proposition 3.3.2 for the unital case, also extends to the present more general setting:

**Proposition 4.1.3** (Homotopy invariance of $K_0$ (general case))**.** *Let $\mathcal{C}$ and $\mathcal{Q}$ be $C^*$-algebras.*

*(i) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homomorphisms, then $K_0(\varphi) = K_0(\psi)$,*

*(ii) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent, then $K_0(\mathcal{C})$ is isomorphic to $K_0(\mathcal{Q})$. More specifically, if* (3.4) *is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ and $K_0(\psi) : K_0(\mathcal{Q}) \to K_0(\mathcal{C})$ are isomorphisms, with $K_0(\varphi)^{-1} = K_0(\psi)$.*

Let us end this section with a construction which will play an important role in the sequel. For any $C^*$-algebra $\mathcal{C}$ one defines *the cone* $C(\mathcal{C})$ and *the suspension* $S(\mathcal{C})$ by

$$C(\mathcal{C}) := \big\{ f \in C([0,1];\mathcal{C}) \mid f(0) = 0 \big\}, \tag{4.2}$$

$$S(\mathcal{C}) := \big\{ f \in C([0,1];\mathcal{C}) \mid f(0) = f(1) = 0 \big\}. \tag{4.3}$$

We have then a short exact sequence

$$0 \longrightarrow S(\mathcal{C}) \overset{\iota}{\hookrightarrow} C(\mathcal{C}) \overset{\pi}{\longrightarrow} \mathcal{C} \longrightarrow 0, \tag{4.4}$$

where $\iota$ is the inclusion mapping, and $\pi(f) = f(1)$ for any $f \in C(\mathcal{C})$.

Note that the cone $C(\mathcal{C})$ is homotopy equivalent to the $C^*$-algebra $\{0\}$. Indeed, for any $t \in [0,1]$ let us define the $*$-homomorphism $\varphi(t) : C(\mathcal{C}) \to C(\mathcal{C})$ by

$$\big[\varphi(t)(f)\big](s) := f(st) \qquad f \in C(\mathcal{C}),\ s \in [0,1].$$

Clearly, the map $[0,1] \ni t \mapsto \big(\varphi(t)\big)(f) \in C(\mathcal{C})$ is continuous, and therefore one has

$$0_{C(\mathcal{C})\to C(\mathcal{C})} = \varphi(0) \sim_h \varphi(1) = \mathrm{id}_{C(\mathcal{C})}.$$

It then easily follows that the $C^*$-algebra $C(\mathcal{C})$ is homotopy equivalent to $\{0\}$, and then from Proposition 4.1.3.(*ii*) and Proposition 4.1.2.(*iii*) that $K_0\big(C(\mathcal{C})\big) = \{0\}$.

## 4.2 The standard picture of the group $K_0$

In Proposition 3.2.4, an explicit formulation of the $K_0$-group for a unital $C^*$-algebra was provided. In this section, we present a similar picture for general $C^*$-algebras. This formulation is very convenient whenever explicit computations involving $K_0$-groups are performed.

Consider an arbitrary $C^*$-algebra $\mathcal{C}$ and the corresponding split exact sequence

$$0 \longrightarrow \mathcal{C} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

One then defines the *scalar mapping* $s$ by

$$s := \lambda \circ \pi : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}},$$

*i.e.* $s(a + \alpha\mathbf{1}) = \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$ and with $\mathbf{1}$ the unit of $\widetilde{\mathcal{C}}$. Note that $\pi\big(s(b)\big) = \pi(b)$ for any $b \in \widetilde{\mathcal{C}}$, and that $b - s(b) \in \mathcal{C}$. As usual, we keep the notation $s$ for the induced $*$-homomorphism $M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{C}})$. Its image is the subset $M_n(\mathbb{C})$ of $M_n(\widetilde{\mathcal{C}})$ consisting of all matrices with scalar entries. For short, any element $a \in M_n(\mathcal{C})$ or $a \in M_n(\widetilde{\mathcal{C}})$ will be called *a scalar element* if $a = s(a)$. On the other hand, note that $a - s(a)$ belongs to $M_n(\mathcal{C})$ for any $a \in M_n(\widetilde{\mathcal{C}})$.

The scalar mapping is natural in the sense that if $\mathcal{C}$ and $\mathcal{Q}$ are $C^*$-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, we then get the commutative diagram:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "s"] \arrow[d, "\tilde{\varphi}"'] & \widetilde{\mathcal{C}} \arrow[d, "\tilde{\varphi}"] \\ \widetilde{\mathcal{Q}} \arrow[r, "s"'] & \widetilde{\mathcal{Q}}. \end{tikzcd} \tag{4.5}$$

The following proposition contains the standard picture of $K_0(\mathcal{C})$:

**Proposition 4.2.1.** *For any $C^*$-algebra $\mathcal{C}$ one has*

$$K_0(\mathcal{C}) = \big\{[p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})\big\}. \tag{4.6}$$

*Moreover, one has*

*(i) For any pair of projections $p, q \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ the following conditions are equivalent:*

*(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,*

*(b) There exist natural numbers $k$ and $\ell$ such that $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_\ell$ in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$,*

*(c) There exist scalar projections $r_1$ and $r_2$ such that $p \oplus r_1 \sim_0 q \oplus r_2$.*

*(ii) If $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ satisfies $[p]_0 - [s(p)]_0 = 0$, then there exists a natural number $m$ with $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$.*

*(iii) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, then*

$$K_0(\varphi)\big([p]_0 - [s(p)]_0\big) = \big[\tilde{\varphi}(p)\big]_0 - \big[s\big(\tilde{\varphi}(p)\big)\big]_0$$

*for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$.*

*Proof.* To prove that equation (4.6) holds, observe first that for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ it follows from the equality $\pi = \pi \circ s$ that

$$K_0(\pi)\big([p]_0 - [s(p)]_0\big) = [\pi(p)]_0 - [(\pi \circ s)(p)]_0 = 0.$$

From it, one infers that $[p]_0 - [s(p)]_0$ belongs to $K_0(\mathcal{C})$ for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$.

Conversely, let $g$ be an arbitrary element of $K_0(\mathcal{C})$, and let $n \in \mathbb{N}^*$ and $p', q' \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ be such that $g = [p']_0 - [q']_0$, see (3.2). Then set

$$p := \begin{pmatrix} p' & 0 \\ 0 & \mathbf{1}_n - q' \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Then one has $p, q \in \mathcal{P}_{2n}(\widetilde{\mathcal{C}})$ and

$$[p]_0 - [q]_0 = [p']_0 + [\mathbf{1}_n - q']_0 - [\mathbf{1}_n]_0 = [p']_0 - [q']_0 = g,$$

where we have used that $[\mathbf{1}_n - q']_0 + [q']_0 = [\mathbf{1}_n]_0$. Since $q = s(q)$ and $K_0(\pi)(g) = 0$ we deduce that

$$[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_0(s)(g) = \big(K_0(\lambda) \circ K_0(\pi)\big)(g) = 0.$$

This shows that $g = [p]_0 - [q]_0 = [p]_0 - [s(p)]_0$.

i) Let $p, q \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ be given, and suppose that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$. Then $[p \oplus s(q)]_0 = [q \oplus s(p)]_0$, and hence $p \oplus s(q) \sim_s q \oplus s(p)$ in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$, by Proposition 3.2.4.$(v)$. By the observations made after Definition 3.2.3, there exists $n \in \mathbb{N}$ such that $p \oplus s(q) \oplus \mathbf{1}_n \sim_0 q \oplus s(p) \oplus \mathbf{1}_n$. This shows that $(a)$ implies $(c)$. To see that $(c)$ implies $(b)$ note that if $r_1$ and $r_2$ are scalar projections in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$ of dimension $k$ and $\ell$, respectively, then $r_1 \sim_0 \mathbf{1}_k$ and $r_2 \sim_0 \mathbf{1}_\ell$ (see Exercise 3.1.4), and hence $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_\ell$.

To see that $(b)$ implies $(a)$ note first that

$$[p \oplus \mathbf{1}_k]_0 - [s(p \oplus \mathbf{1}_k)]_0 = [p]_0 + [\mathbf{1}_k]_0 - [s(p)]_0 - [\mathbf{1}_k]_0 = [p]_0 - [s(p)]_0.$$

Therefore, it is sufficient to show that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ when $p \sim_0 q$. Suppose accordingly that $p = v^*v$ and $q = vv^*$ for some partial isometry $v \in M_{m,n}(\widetilde{\mathcal{C}})$. Let $s(v) \in M_{m,n}(\mathbb{C})$, viewed as a subset of $M_{m,n}(\widetilde{\mathcal{C}})$, be the matrix obtained by applying $s$ to each entry of $v$. Then $s(v)^*s(v) = s(p)$ and $s(v)s(v)^* = s(q)$, and so $s(p) \sim_0 s(q)$. As a consequence, $[p]_0 = [q]_0$ and $[s(p)]_0 = [s(q)]_0$, and this proves that $(a)$ holds.

ii) If $[p]_0 - [s(p)]_0 = 0$, then $p \sim_s s(p)$ by Proposition 3.2.4.$(v)$, and there exists $m \in \mathbb{N}$ such that $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$, see the observations made juste before Proposition 3.2.4. Note that $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$ is equivalent to $p \oplus \mathbf{1}_m \sim_0 s(p) \oplus \mathbf{1}_m$ since $p$ and $s(p)$ belong to the same matrix algebra over $\widetilde{\mathcal{C}}$.

iii) By definition one has

$$\begin{aligned} K_0(\varphi)\big([p]_0 - [s(p)]_0\big) &= K_0(\tilde{\varphi})\big([p]_0 - [s(p)]_0\big) \\ &= \big[\tilde{\varphi}(p)\big]_0 - \big[\tilde{\varphi}\big(s(p)\big)\big]_0 = \big[\tilde{\varphi}(p)\big]_0 - \big[s\big(\tilde{\varphi}(p)\big)\big]_0. \end{aligned}$$

□

The following slightly technical statement will be used in the next section. If proof is provided in [RLL00, Lem. 4.2.3].

**Lemma 4.2.2.** *Let $\mathcal{C}$, $\mathcal{Q}$ be $C^*$-algebras, and $\varphi : \mathcal{C} \to \mathcal{Q}$ a $*$-homomorphism. Let also $g$ be an element of $K_0(\mathcal{C})$ which belongs to the kernel of $K_0(\varphi)$. Then:*

*(i) There exist $n \in \mathbb{N}^*$, $p \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ and a unitary element $u \in M_n(\widetilde{\mathcal{Q}})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s\big(\tilde{\varphi}(p)\big)$.*

*(ii) If $\varphi$ is surjective, one can choose $u = \mathbf{1}$ in the point $(i)$.*

## 4.3 Half and split exactness and stability of $K_0$

Let us start this section with an easy lemma which described what happens when a unit is added to a short exact sequence. The proof of this lemma is left as an exercise.

**Lemma 4.3.1.** *Consider the short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

*and let $n \in \mathbb{N}^*$. Then*

*(i) The map $\tilde{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\widetilde{\mathcal{C}})$ is injective,*

*(ii) An element $a \in M_n(\widetilde{\mathcal{C}})$ belongs to $\mathsf{Ran}(\tilde{\varphi})$ if and only if $\tilde{\psi}(a) = s\big(\tilde{\psi}(a)\big)$, with $s : \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{Q}}$ the scalar mapping.*

**Proposition 4.3.2** (Half exactness of $K_0$)**.** *Every short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

*induces an exact sequence of Abelian groups*

$$K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \xrightarrow{K_0(\psi)} K_0(\mathcal{Q}),$$

*that is* $\mathsf{Ran}\big(K_0(\varphi)\big) = \mathsf{Ker}\big(K_0(\psi)\big)$.

*Proof.* By functoriality of $K_0$ one already knows that

$$K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = K_0(0_{\mathcal{J}\to\mathcal{Q}}) = 0_{K_0(\mathcal{J})\to K_0(\mathcal{Q})},$$

which implies that $\mathsf{Ran}\big(K_0(\varphi)\big) \subset \mathsf{Ker}\big(K_0(\psi)\big)$.

Conversely, assume that $g \in \mathsf{Ker}\big(K_0(\psi)\big)$. According to Lemma 4.2.2.$(ii)$ there exist $n \in \mathbb{N}^*$ and $p \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\psi}(p) = s\big(\tilde{\psi}(p)\big)$. Then by Lemma 4.3.1.$(ii)$ there exists $e \in M_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(e) = p$. Since by Lemma 4.3.1.$(i)$ the map $\tilde{\varphi}$ is injective, one infers that $e \in \mathcal{P}_n(\widetilde{\mathcal{J}})$. Therefore,

$$g = [\tilde{\varphi}(e)]_0 - \big[s\big(\tilde{\varphi}(e)\big)\big]_0 = \tilde{\varphi}\big([p]_0 - [s(p)]_0\big) = K_0(\varphi)\big([e]_0 - [s(e)]_0\big) \tag{4.7}$$

which thus belongs to $\mathsf{Ran}\big(K_0(\varphi)\big)$. Note that the standard picture of $K_0(\mathcal{J})$ has been used in the last equality of (4.7). These two inclusions lead to the statement. $\square$

**Proposition 4.3.3** (Split exactness of $K_0$)**.** *Every split exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \mathcal{Q} \longrightarrow 0$$

*induces a split exact sequence of Abelian groups*

$$0 \longrightarrow K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(\mathcal{Q}) \longrightarrow 0.$$

*Proof.* It follows from Proposition 4.3.2 that the equality $\mathsf{Ran}\big(K_0(\varphi)\big) = \mathsf{Ker}\big(K_0(\psi)\big)$ holds. In addition, from the functoriality of $K_0$ one infers that

$$\mathrm{id}_{K_0(\mathcal{Q})} = K_0(\mathrm{id}_{\mathcal{Q}}) = K_0(\psi) \circ K_0(\lambda)$$

which implies that $K_0(\psi)$ is surjective and the splitness of the sequence. As a consequence, it only remains to show that $K_0(\varphi)$ is injective.

For the injectivity, let us consider $g \in \mathsf{Ker}\big(K_0(\varphi)\big)$. By Lemma 4.2.2.$(i)$, there exist $n \in \mathbb{N}^*$, $p \in \mathcal{P}_n(\widetilde{\mathcal{J}})$ and a unitary element $u \in M_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s\big(\tilde{\varphi}(p)\big)$. Set $v := (\tilde{\lambda} \circ \tilde{\psi})(u^*)u$, and observe that $v$ is a unitary element of $M_n(\tilde{\mathcal{C}})$ and $\tilde{\psi}(v) = \mathbf{1}$. By Lemma 4.3.1.$(ii)$ there exists an element $w \in M_n(\widetilde{\mathcal{J}})$ with $\tilde{\varphi}(w) = v$. In addition, since $\tilde{\varphi}$ is injective, $w$ must be unitary. Then, from the computation (use Lemma 4.3.1.$(ii)$ in the second last equality)

$$\begin{aligned}\tilde{\varphi}(wpw^*) &= v\tilde{\varphi}(p)v^* = (\tilde{\lambda} \circ \tilde{\psi})(u^*)s\big(\tilde{\varphi}(p)\big)(\tilde{\lambda} \circ \tilde{\psi})(u) \\ &= (\tilde{\lambda} \circ \tilde{\psi})\big(u^*s\big(\tilde{\varphi}(p)\big)u\big)) = (\tilde{\lambda} \circ \tilde{\psi})\big(\tilde{\varphi}(p)\big) = s\big(\tilde{\varphi}(p)\big) = \tilde{\varphi}\big(s(p)\big)\end{aligned}$$

and by the injectivity of $\tilde{\varphi}$ we conclude that $wpw^* = s(p)$. This shows that $p \sim_u s(p)$ in $M_n(\widetilde{\mathcal{J}})$, and hence that $g = 0$. □

Let us study the behavior of $K_0$ with respect to direct sum of $C^*$-algebras.

**Proposition 4.3.4.** *For any $C^*$-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ the $K_0$-groups $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2)$ and $K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$ are isomorphic.*

*Proof.* For $i \in \{1, 2\}$, recall that $\iota_i : \mathcal{C}_i \to \mathcal{C}_1 \oplus \mathcal{C}_2$ denotes the canonical inclusion $*$-homomorphism (already introduced in Section 1.1) and let us set $\pi_i : \mathcal{C}_1 \oplus \mathcal{C}_2 \to \mathcal{C}_i$ for the projection $*$-homomorphism. The sequence

$$0 \longrightarrow \mathcal{C}_1 \xrightarrow{\iota_1} \mathcal{C}_1 \oplus \mathcal{C}_2 \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} \mathcal{C}_2 \longrightarrow 0,$$

is a split exact short exact sequence of $C^*$-algebras, and therefore by Proposition 4.3.3 one directly infers that

$$0 \longrightarrow K_0(\mathcal{C}_1) \xrightarrow{K_0(\iota_1)} K_0(\mathcal{C}_1 \oplus \mathcal{C}_2) \underset{K_0(\iota_2)}{\overset{K_0(\pi_2)}{\rightleftarrows}} K_0(\mathcal{C}_2) \longrightarrow 0$$

is a split exact short exact sequence. It then follows by a standard argument (five lemma) that $K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$ is isomorphic to $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2)$, with the isomorphism given by

$$K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2) \ni (g,h) \mapsto K_0(\iota_1)(g) + K_0(\iota_2)(h) \in K_0(\mathcal{C}_1 \oplus \mathcal{C}_2).$$

□

We shall now see on two examples that the functor $K_0$ is not exact. Note that it would be the case if any short exact sequence of $C^*$-algebras would be transformed in a short exact sequence at the level of the $K_0$-groups.

**Example 4.3.5.** *Consider the exact sequence*

$$0 \longrightarrow C_0\big((0,1)\big) \overset{\iota}{\hookrightarrow} C\big([0,1]\big) \overset{\psi}{\longrightarrow} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

*One deduces from Proposition 4.3.4 and from Example 3.4.1 that $K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$, and from Example 3.4.3 that $K_0\big(C\big([0,1]\big)\big) \cong \mathbb{Z}$. Therefore $K_0(\psi)$ can not be surjective.*

**Example 4.3.6.** *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and consider the $C^*$-algebra $\mathcal{B}(\mathcal{H})$. The $C^*$-subalgebra $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ is an ideal of $\mathcal{B}(\mathcal{H})$, and the quotient algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called* the Calkin algebra. *Thus we have a short exact sequence*

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \overset{\iota}{\hookrightarrow} \mathcal{B}(\mathcal{H}) \overset{\psi}{\longrightarrow} \mathcal{Q}(\mathcal{H}) \longrightarrow 0.$$

*From Example 3.4.2 one knows that $K_0\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$. It will be shown later on that $K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$ which means that $K_0(\iota)$ can not be injective.*

We finally state an important result for the computation of the $K_0$-groups for $C^*$-algebras, but refer to [RLL00, Prop. 4.3.8 & 6.4.1] for its proof.

**Proposition 4.3.7** (Stability of $K_0$)**.** *Let $\mathcal{C}$ be a $C^*$-algebra and let $n \in \mathbb{N}^*$. Then $K_0(\mathcal{C})$ is isomorphic to $K_0\big(M_n(\mathcal{C})\big)$. In addition, for any separable Hilbert space $\mathcal{H}$ the following equality holds*

$$K_0\big(\mathcal{C} \otimes \mathcal{K}(\mathcal{H})\big) \cong K_0(\mathcal{C}).$$

**Extension 4.3.8.** *Work on the notion of ordered Abelian $K_0$-group, as presented for example in [RLL00, Sec. 5.1].*

**Extension 4.3.9.** *Work on the irrational rotation $C^*$-algebra, as introduced in Exercise 5.8 of [RLL00]. This algebra has played an important role in operator algebra, and the literature on the subject is very rich.*

**Extension 4.3.10.** *Work on the notion of inductive limit of $C^*$-algebras, as presented in Chapter 6 of [RLL00], and more precisely in Section 6.2 of this reference.*

# Chapter 5

# The functor $K_1$

In this chapter, we define the $K_1$-group of a $C^*$-algebra $\mathcal{C}$ as the set of homotopy equivalent classes of unitary elements in the matrix algebras over $\widetilde{\mathcal{C}}$. It will also be shown that the functor $K_1$ is half exact and homotopy invariant. Since we shall prove in the sequel that $K_1(\mathcal{C})$ is naturally isomorphic to $K_0\big(S(\mathcal{C})\big)$, some of the properties of $K_1$ will directly be inferred from equivalent properties of $K_0$. For that reason, their proofs will be provided only once this isomorphism has been exhibited.

## 5.1 Definition of the $K_1$-group

Let us first recall that the set of unitary elements of a unital $C^*$-algebra $\mathcal{C}$ is denoted by $\mathcal{U}(\mathcal{C})$. For any $n \in \mathbb{N}^*$ one sets

$$\mathcal{U}_n(\mathcal{C}) := \mathcal{U}\big(M_n(\mathcal{C})\big) \qquad \text{and} \qquad \mathcal{U}_\infty(\mathcal{C}) := \bigcup_{n\in\mathbb{N}^*} \mathcal{U}_n(\mathcal{C}).$$

We define a binary operation $\oplus$ on $\mathcal{U}_\infty(\mathcal{C})$: for $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$ one sets

$$u \oplus v := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(\mathcal{C}).$$

In addition, a relation $\sim_1$ on $\mathcal{U}_\infty(\mathcal{C})$ is defined as follows: for $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$ one writes $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus \mathbf{1}_{k-n} \sim_h v \oplus \mathbf{1}_{k-m}$ in $\mathcal{U}_k(\mathcal{C})$. With these definitions at hand one can show:

**Lemma 5.1.1.** *Let $\mathcal{C}$ be a unital $C^*$-algebra. Then:*

*(i)* $\sim_1$ *is an equivalence relation on* $\mathcal{U}_\infty(\mathcal{C})$,

*(ii)* $u \sim_1 u \oplus \mathbf{1}_n$ *for any* $u \in \mathcal{U}_\infty(\mathcal{C})$ *and* $n \in \mathbb{N}$,

*(iii)* $u \oplus v \sim_1 v \oplus u$ *for any* $u, v \in \mathcal{U}_\infty(\mathcal{C})$,

*(iv) If* $u, v, u', v' \in \mathcal{U}_\infty(\mathcal{C})$, $u \sim_1 u'$ *and* $v \sim_1 v'$ *then* $u \oplus v \sim_1 u' \oplus v'$,

*(v) If $u, v \in \mathcal{U}_n(\mathcal{C})$, then $uv \sim_1 vu \sim_1 u \oplus v$,*

*(vi) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ for any $u, v, w \in \mathcal{U}_\infty(\mathcal{C})$.*

*Proof.* The proofs of $(i), (ii)$ and $(vi)$ are trivial, and $(v)$ follows from Lemma 2.1.4. For the proof of $(iii)$, let us consider $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$, and set

$$z = \begin{pmatrix} 0 & \mathbf{1}_m \\ \mathbf{1}_n & 0 \end{pmatrix} \in \mathcal{U}_{n+m}(\mathcal{C}).$$

Then by taking $(v)$ into account, one gets

$$v \oplus u = z(u \oplus v)z^* \sim_1 z^* z(u \oplus v) = u \oplus v.$$

For the proof of $(iv)$ it is sufficient to show that

(I) $(u \oplus \mathbf{1}_k) \oplus (v \oplus \mathbf{1}_\ell) \sim_1 u \oplus v$ for any $u, v \in \mathcal{U}_\infty(\mathcal{C})$ and any $k, \ell \in \mathbb{N}$,

(II) $u \sim_h u'$ and $v \sim_h v'$ imply that $u \oplus v \sim_h u' \oplus v'$ for all $u, u' \in \mathcal{U}_n(\mathcal{C})$ and $v, v' \in \mathcal{U}_m(\mathcal{C})$.

Now, statement (I) follows from $(ii)$, $(iii)$ and $(vi)$. To see that (II) holds, let $t \mapsto u(t)$ and $t \mapsto v(t)$ be continuous paths of unitary elements with $u = u(0)$, $u' = u(1)$, $v = v(0)$ and $v' = v(1)$. Then $t \mapsto u(t) \oplus v(t)$ is a continuous path of unitary elements from $u \oplus v$ to $u' \oplus v'$. □

**Definition 5.1.2.** *For any $C^*$-algebra $\mathbb{C}$ one defines*

$$K_1(\mathcal{C}) := \mathcal{U}_\infty(\widetilde{\mathcal{C}}) / \sim_1 .$$

*The equivalent class in $K_1(\mathcal{C})$ containing $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ is denoted by $[u]_1$. A binary operation on $K_1(\mathcal{C})$ is defined by $[u]_1 + [v]_1 := [u \oplus v]_1$ for any $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.*

It follows from Lemma 5.1.1 that $+$ is well-defined, commutative, associative, has zero element $[\mathbf{1}]_1 \equiv [\mathbf{1}_n]_1$ for any $n \in \mathbb{N}^*$, and that

$$0 = [\mathbf{1}]_1 = [uu^*]_1 = [u]_1 + [u^*]_1$$

for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$. All this shows that $\big(K_1(\mathcal{C}), +\big)$ is an Abelian group, and that $-[u]_1 = [u^*]_1$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

We now collect these information and provide the standard picture of $K_1$. The statements follow either directly from the definitions or from Lemma 5.1.1.

**Proposition 5.1.3.** *Let $\mathcal{C}$ be a $C^*$-algebra. Then*

$$K_1(\mathcal{C}) = \{[u]_1 \mid u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})\},$$

*and the map $[\cdot]_1 : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to K_1(\mathcal{C})$ has the following properties:*

*(i)* $[u \oplus v]_1 = [u]_1 + [v]_1$ *for any* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$

*(ii)* $[\mathbf{1}]_1 = 0$,

*(iii)* *If* $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ *and* $u \sim_h v$, *then* $[u]_1 = [v]_1$,

*(iv)* *If* $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$, *then* $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$,

*(v)* *For* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$, $[u]_1 = [v]_1$ *if and only if* $u \sim_1 v$.

We provide some additional information on the $K_1$-group. The first one corresponds to the universal property of $K_1$, which is the analogue of Proposition 3.2.5 for $K_0$.

**Proposition 5.1.4** (Universal property of $K_1$)**.** *Let* $\mathcal{C}$ *be a* $C^*$*-algebra and let* $H$ *be an Abelian group. Suppose that there exists* $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to H$ *satisfying the three conditions:*

*(i)* $\nu(u \oplus v) = \nu(u) + \nu(v)$ *for any* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

*(ii)* $\nu(\mathbf{1}) = 0$,

*(iii)* *If* $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ *for some* $n \in \mathbb{N}^*$ *and if* $u \sim_h v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$, *then* $\nu(u) = \nu(v)$.

*Then there exists a unique group homomorphism* $\alpha : K_1(\mathcal{C}) \to H$ *such that the diagram*

$$\begin{tikzcd} \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \arrow[d, "{[\cdot]_1}"'] \arrow[dr, "\nu"] & \\ K_1(\mathcal{C}) \arrow[r, "\alpha"'] & H \end{tikzcd} \tag{5.1}$$

*is commutative.*

*Proof.* We first show that if $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and $v \in \mathcal{U}_m(\widetilde{\mathcal{C}})$ satisfies $u \sim_1 v$, then $\nu(u) = \nu(v)$. For that purpose, let $k \in \mathbb{N}$ with $k \geq \max\{m, n\}$ such that $u \oplus \mathbf{1}_{k-n} \sim_h v \oplus \mathbf{1}_{k-m}$ in $\mathcal{U}_k(\widetilde{\mathcal{C}})$. By taking $(i)$ and $(ii)$ into accounts, one infers that $\nu(\mathbf{1}_r) = 0$ for any $r \in \mathbb{N}^*$. As a consequence, $(i)$ and $(iii)$ imply that

$$\nu(u) = \nu(u \oplus \mathbf{1}_{k-n}) = \nu(v \oplus \mathbf{1}_{k-m}) = \nu(v).$$

It follows from this equality that there exists a map $\alpha : K_1(A) \to H$ making the diagram (5.1) commutative. Then, the computation

$$\alpha\big([u]_1 + [v]_1\big) = \alpha([u \oplus v]_1) = \nu(u \oplus v) = \nu(u) + \nu(v) = \alpha([u]_1) + \alpha([v]_1)$$

shows that $\alpha$ is a group morphism. The uniqueness of $\alpha$ follows from the surjectivity of the map $[\cdot]_1$. $\square$

If $\mathcal{C}$ is a unital algebra, it would be natural to define directly the $K_1$-group of $\mathcal{C}$ by $\mathcal{U}_\infty(\mathcal{C})/\sim_1$ without using the algebra $\widetilde{\mathcal{C}}$. This is indeed possible, as shown in the following statement. For that purpose, recall from the proof of Lemma 2.2.4 that if $\tilde{\mathbf{1}}$ denotes the unit of $\widetilde{\mathcal{C}}$ and if $\mathbf{1}$ denotes the unit of $\mathcal{C}$, then $1 := \tilde{\mathbf{1}} - \mathbf{1}$ is a projection in $\widetilde{\mathcal{C}}$. In addition, $\widetilde{\mathcal{C}} = \mathcal{C} + \mathbb{C}1$, with $a1 = 1a = 0$ for any $a \in \mathcal{C}$. One also defines the $*$-homomorphism $\mu : \widetilde{\mathcal{C}} \to \mathcal{C}$ by $\mu(a+\alpha 1) := a$ and extends it to a unital $*$-homomorphism $M_n(\widetilde{\mathcal{C}}) \to M_n(\mathcal{C})$ for any $n \in \mathbb{N}^*$. In this way one obtains a map $\mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\mathcal{C})$.

**Proposition 5.1.5.** *Let $\mathcal{C}$ be a unital $C^*$-algebra. Then there exists an isomorphism $\rho : K_1(\mathcal{C}) \to \mathcal{U}_\infty(\mathcal{C})/\sim_1$ making the following diagram commutative:*

$$\begin{tikzcd} \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] & \mathcal{U}_\infty(\mathcal{C}) \arrow[d] \\ K_1(\mathcal{C}) \arrow[r, "\rho"'] & \mathcal{U}_\infty(\mathcal{C})/\sim_1 . \end{tikzcd}$$

*Proof.* Observe first that the map $\mu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\mathcal{C})$ is surjective. Then, it is sufficient to show that

(I) $\mu(u) \sim_1 \mu(v)$ if and only if $u \sim_1 v$ for any $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

(II) $\mu(u \oplus v) = \mu(u) \oplus \mu(v)$ for any $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

Clearly, (II) is a direct consequence of the definition of the map $\mu$. For (I) it is sufficient to show that

(I') $\mu(u) \sim_h \mu(v)$ in $\mathcal{U}_n(\mathcal{C})$ if and only if $u \sim_h v$ in $\mathcal{U}_n(\widetilde{\mathcal{C}})$, for any $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and any $n \in \mathbb{N}^*$.

For that purpose, observe that if $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ are such that $u \sim_h v$, then $\mu(u) \sim_h \mu(v)$. For the converse implication, assume that $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and that $\mu(u) \sim_h \mu(v)$ in $\mathcal{U}_n(\mathcal{C})$. By the definition of $\mu$ one can find $u_0$ and $v_0$ in $\mathcal{U}_n(\mathbb{C}1)$ such that $u = \mu(u) + u_0$ and $v = \mu(v) + v_0$. By Corollary 2.1.3 one infers that $u_0 \sim_h v_0$ in $M_n(\mathbb{C}1)$, which easily proves that $u \sim_h v$ in $M_n(\widetilde{\mathcal{C}})$. Indeed, one can consider the continuous path $t \mapsto a(t)$ and $t \mapsto b(t)$ of unitary elements in $M_n(\mathcal{C})$ and $M_n(\mathbb{C}1)$, respectively, with $\mu(u) = a(0)$, $\mu(v) = a(1)$, $u_0 = b(0)$ and $u_1 = b(1)$. Then $t \mapsto a(t) + b(t)$ is a continuous path in $\mathcal{U}_n(\widetilde{\mathcal{C}})$ with $u = a(0) + b(0)$ and $v = a(1) + b(1)$. $\square$

When $\mathcal{C}$ is unital, we shall often identify $K_1(\mathcal{C})$ with $\mathcal{U}_\infty(\mathcal{C})/\sim_1$ through the isomorphism $\rho$ of the previous proposition. If $u$ is a unitary element of $\mathcal{U}_\infty(\mathcal{C})$, then $[u]_1$ will denote the element of $K_1(\mathcal{C})$ it represents under this identification. As a immediate consequence of the previous proposition, one also obtains that for any $C^*$-algebra:

$$K_1(\mathcal{C}) \cong K_1(\widetilde{\mathcal{C}}). \tag{5.2}$$

Let us finally conclude this section with the explicit computation of a $K_1$-group.

**Lemma 5.1.6.** *One has $K_1(\mathbb{C}) = K_1\big(M_n(\mathbb{C})\big) = \{0\}$ for any $n \in \mathbb{N}^*$. More generally one has $K_1\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$ for any separable Hilbert space $\mathcal{H}$.*

*Proof.* It has been proved in Corollary 2.1.3 that the unitary group of $M_k\big(M_n(\mathbb{C})\big) = M_{kn}(\mathbb{C})$ is connected for every $n$ and $k$ in $\mathbb{N}^*$. This implies that $\mathcal{U}_\infty\big(M_n(\mathbb{C})\big)/\sim_1$ is the trivial group with only one element. From the description of $K_1$ for a unital $C^*$-algebra provided by Proposition 5.1.5 one infers that $K_1\big(M_n(\mathbb{C})\big) = \{0\}$.

Let us now consider any separable Hilbert space $\mathcal{H}$ and first show that $u \sim_h \mathbf{1}_n$ for any unitary element $u \in M_n\big(\mathcal{B}(\mathcal{H})\big)$. Indeed, let us define $\varphi : \mathbb{T} \to [0, 2\pi)$ by

$$\varphi(e^{i\theta}) = \theta, \qquad 0 \leq \theta < 2\pi.$$

Then $\varphi$ is a bounded Borel measurable map, and $z = e^{i\varphi(z)}$ for any $z \in \mathbb{T}$. As a consequence, for any $u \in \mathcal{U}_n\big(\mathcal{B}(\mathcal{H})\big) = \mathcal{U}\big(\mathcal{B}(\mathcal{H}^n)\big)$, one infers that $\varphi(u) = \varphi(u)^*$ in $\mathcal{B}(\mathcal{H}^n)$, and that $u = e^{i\varphi(u)}$. By Lemma 2.1.2.$(i)$ it follows that $u \sim_h \mathbf{1}_n$. Consequently, one deduces that $u \sim_1 \mathbf{1}$, and then that $\mathcal{U}_\infty\big(\mathcal{B}(\mathcal{H})\big)/\sim_1 = \{0\}$. In other words, one concludes that $K_1\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$ as above. □

## 5.2 Functoriality of $K_1$

This section is partially analogue to Section 3.3. Let us first consider two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a $*$-homomorphism. Then $\varphi$ induces a unital $*$-homomorphism $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ which itself extends to a unital $*$-homomorphism $\tilde{\varphi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ for any $n \in \mathbb{N}^*$. This gives rise to a map $\tilde{\varphi} : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$, and one can set $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to K_1(\mathcal{Q})$ by $\nu(u) := [\tilde{\varphi}(u)]_1$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$. It is straightforward to check that $\nu$ satisfies the three conditions of Proposition 5.1.4, and hence there exists precisely one group homomorphism $K_1(\varphi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$ with the property

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1 \tag{5.3}$$

for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

Note that if $\mathcal{C}$ and $\mathcal{Q}$ are unital $C^*$-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a unital $*$-homomorphism, then $K_1(\varphi)([u]_1) = [\varphi(u)]_1$ for any $u \in \mathcal{U}_\infty(\mathcal{C})$.

The following proposition shows that $K_1$ is a homotopy invariant functor which preserves the zero objects.

**Proposition 5.2.1** (Functoriality and homotopy invariance of $K_1$)**.** *Let $\mathcal{J}$, $\mathcal{C}$ and $\mathcal{Q}$ be $C^*$-algebras. Then*

*(i)* $K_1(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{K_1(\mathcal{C})}$,

*(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are $*$-homomorphisms, then*

$$K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi),$$

*(iii)* $K_1(\{0\}) = \{0\}$,

*(iv)* $K_1(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_1(\mathcal{C})\to K_1(\mathcal{Q})}$,

*(v) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homomorphisms, then $K_1(\varphi) = K_1(\psi)$,*

*(vi) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent, then $K_1(\mathcal{C})$ is isomorphic to $K_1(\mathcal{Q})$. More specifically, if (3.4) is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then $K_1(\varphi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$ and $K_1(\psi) : K_1(\mathcal{Q}) \to K_1(\mathcal{C})$ are isomorphisms, with $K_1(\varphi)^{-1} = K_1(\psi)$.*

*Proof.* The proof of $(i)$ and $(ii)$ can directly be inferred from (5.3) together with the equalities $\widetilde{\mathrm{id}_{\mathcal{C}}} = \mathrm{id}_{\widetilde{\mathcal{C}}}$ and $\widetilde{(\psi \circ \varphi)} = \tilde{\psi} \circ \tilde{\varphi}$.

As already mentioned in (5.2), the equality $K_1(\mathcal{C}) = K_1(\widetilde{\mathcal{C}})$ holds for any $C^*$-algebra. In particular, $K_1(\{0\})$ is isomorphic to $K_1(\mathbb{C})$, which is equal to $\{0\}$ by Lemma 5.1.6. This implies $(iii)$.

The zero homomorphism $0_{\mathcal{C}\to\mathcal{Q}}$ can be seen as the composition of the maps $\mathcal{C} \to \{0\}$ and $\{0\} \to \mathcal{Q}$. Hence, $(iv)$ follows from $(iii)$ and $(ii)$.

(v) Let us now consider a path $t \mapsto \varphi(t)$ of $*$-homomorphisms from $\mathcal{C}$ to $\mathcal{Q}$, with $\varphi(0) = \varphi$ and $\varphi(1) = \psi$, and such that the map $[0,1] \ni t \mapsto \varphi(t)(a) \in \mathcal{Q}$ is continuous, for any $a \in \mathcal{C}$. The induced $*$-homomorphism $\tilde{\varphi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ is unital, for any $n \in \mathbb{N}^*$, and the map $[0,1] \ni t \mapsto \varphi(t)(a) \in M_n(\widetilde{\mathcal{Q}})$ is continuous, for any $a \in M_n(\widetilde{\mathcal{C}})$. Hence for any $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ one has in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$:

$$\tilde{\varphi}(u) = \tilde{\varphi}(0)(u) \sim_h \tilde{\varphi}(1)(u) = \tilde{\psi}(u).$$

As a consequence, one infers that

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1 = [\tilde{\psi}(u)]_1 = K_1(\psi)([u]_1),$$

which proves $(v)$.

Finally, statement $(vi)$ is a consequence of $(i)$, $(ii)$ and $(v)$. □

Let us also prove a short lemma which will be useful in the next proposition.

**Lemma 5.2.2.** *Let $\mathcal{C}$ and $\mathcal{Q}$ be $C^*$-algebras, let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a $*$-homomorphism, and let $g \in \mathsf{Ker}\big(K_1(\varphi)\big)$. Then*

*(i) There exists an element $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ such that $g = [u]_1$ and $\tilde{\varphi}(u) \sim_h \mathbf{1}$,*

*(ii) If $\varphi$ is surjective, then there exists $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ such that $g = [u]_1$ and $\tilde{\varphi}(u) = \mathbf{1}$.*

*Proof.* (i) Choose $v \in \mathcal{U}_m(\widetilde{\mathcal{C}})$ such that $g = [v]_1$. Then $[\tilde{\varphi}(v)]_1 = 0 = [\mathbf{1}_m]_1$, and hence there exists an integer $n \geq m$ such that

$$\tilde{\varphi}(v) \oplus \mathbf{1}_{n-m} \sim_h \mathbf{1}_m \oplus \mathbf{1}_{n-m} = \mathbf{1}_n.$$

Set $u = v \oplus \mathbf{1}_{n-m}$, and then $[u]_1 = [v]_1 = g$ and $\tilde{\varphi}(u) = \tilde{\varphi}(v) \oplus \mathbf{1}_{n-m} \sim_h \mathbf{1}_n$.

(ii) Use $(i)$ to find $v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $g = [v]_1$ and $\tilde{\varphi}(v) \sim_h \mathbf{1}$. By Lemma 2.1.7.$(iii)$ and $(i)$, there exists $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $\tilde{\varphi}(w) = \tilde{\varphi}(v)$ and $w \sim_h \mathbf{1}$. Then $u := w^*v$ has the desired properties. □

**Proposition 5.2.3** (Half exactness of $K_1$)**.** *Every short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

*induces an exact sequence of Abelian groups*

$$K_1(\mathcal{J}) \xrightarrow{K_1(\varphi)} K_1(\mathcal{C}) \xrightarrow{K_1(\psi)} K_1(\mathcal{Q}),$$

*that is* $\mathsf{Ran}\big(K_1(\varphi)\big) = \mathsf{Ker}\big(K_1(\psi)\big)$.

*Proof.* By functoriality of $K_1$ one already knows that

$$K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi) = K_1(0_{\mathcal{J}\to\mathcal{Q}}) = 0_{K_1(\mathcal{J})\to K_1(\mathcal{Q})},$$

which implies that $\mathsf{Ran}\big(K_1(\varphi)\big) \subset \mathsf{Ker}\big(K_1(\psi)\big)$.

Conversely, assume that $g \in \mathsf{Ker}\big(K_1(\psi)\big)$. According to Lemma 5.2.2.$(ii)$ there exist $n \in \mathbb{N}^*$ and $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [u]_1$ and $\tilde{\psi}(u) = \mathbf{1}$. Then, by Lemma 4.3.1.$(ii)$ there exists $v \in M_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(v) = u$. Finally, $[v]_1$ belongs to $K_1(\mathcal{J})$, and $K_1(\varphi)([v]) = [\tilde{\varphi}(v)]_1 = [u]_1 = g$. □

Let us now mention that the functor $K_1$ is split exact and preserves direct sums of $C^*$-algebras. These statements can be proved in the same way as for the functor $K_0$ in Propositions 4.3.3 and 4.3.4. These statements also follow from the isomorphism $K_1(\mathcal{C}) \cong K_0\big(S(\mathcal{C})\big)$ which will be established later on. For this reason, we state these results without providing a proof.

**Proposition 5.2.4** (Split exactness of $K_1$)**.** *Every split exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \mathcal{Q} \longrightarrow 0$$

*induces a split exact sequence of Abelian groups*

$$0 \longrightarrow K_1(\mathcal{J}) \xrightarrow{K_1(\varphi)} K_1(\mathcal{C}) \underset{K_1(\lambda)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(\mathcal{Q}) \longrightarrow 0.$$

**Proposition 5.2.5.** *For any $C^*$-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ the $K_1$-groups $K_1(\mathcal{C}_1 \oplus \mathcal{C}_2)$ and $K_1(\mathcal{C}_1) \oplus K_1(\mathcal{C}_2)$ are isomorphic. More precisely, if $\iota_i : \mathcal{C}_i \to \mathcal{C}_1 \oplus \mathcal{C}_2$ denotes the canonical inclusion $*$-homomorphism, then the group morphism is provided by the map*

$$K_1(\mathcal{C}_1) \oplus K_1(\mathcal{C}_2) \ni (g, h) \mapsto K_1(\iota_1)(g) + K_1(\iota_2)(h) \in K_1(\mathcal{C}_1 \oplus \mathcal{C}_2).$$

We close this section with an important result for the computation of $K_1$-groups, which is the analogue for $K_1$ of the content of Proposition 4.3.7 on the stability of $K_0$. Note that the proof of the following statement can be proved from its analogue for $K_0$ by taking the isomorphism $K_1(\mathcal{C}) \cong K_0\big(S(\mathcal{C})\big)$ into account.

**Proposition 5.2.6** (Stability of $K_1$)**.** *Let $\mathcal{C}$ be a $C^*$-algebra and let $n \in \mathbb{N}^*$. Then $K_1(\mathcal{C})$ is isomorphic to $K_1\big(M_n(\mathcal{C})\big)$. In addition, for any separable Hilbert space $\mathcal{H}$ the following equality holds*

$$K_1\big(\mathcal{C} \otimes \mathcal{K}(\mathcal{H})\big) \cong K_1(\mathcal{C}). \tag{5.4}$$

**Corollary 5.2.7.** *For any separable Hilbert space $\mathcal{H}$ one has $K_1\big(\mathcal{K}(\mathcal{H})\big) = \{0\}$.*

*Proof.* From equation (5.4) one infers that $K_1\big(\mathcal{K}(\mathcal{H})\big) \cong K_1(\mathbb{C})$, but $K_1(\mathbb{C}) = \{0\}$ by Lemma 5.1.6. □

**Extension 5.2.8.** *Work on the relations between $K_1$-group and determinant for unital Abelian $C^*$-algebras, as presented in [RLL00, Sec. 8.3].*

# Chapter 6

# The index map

In this chapter, we introduce the index map associated with the short exact sequence

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0 \tag{6.1}$$

of $C^*$-algebras. This map is a group homomorphism $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ that gives rise to an exact sequence

$$\begin{tikzcd} K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\ K_0(\mathcal{Q}) & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"] \end{tikzcd} . \tag{6.2}$$

The index map generalizes the classical Fredholm index of Fredholm operators on a Hilbert space.

## 6.1 Definition of the index map

Before introducing the index map, two preliminary lemmas are necessary.

**Lemma 6.1.1.** *Consider the short exact sequence* (6.1) *and let* $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$.

*(i) There exist* $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ *and* $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ *such that*

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad s(p) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix},$$

*(ii) If* $v$ *and* $p$ *are as in* (i) *and if* $w \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ *and* $q \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ *satisfy*

$$\tilde{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(q) = w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*,$$

*then* $s(q) = \operatorname{diag}(\mathbf{1}_n, 0_n)$ *and* $p \sim_u q$ *in* $\mathcal{P}_{2n}(\widetilde{\mathcal{J}})$.

*Proof.* (i) Since the unital $*$-homomorphism $\tilde{\psi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ is surjective, it follows from Lemma 2.1.7.$(ii)$ that there exists $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ such that $\tilde{\psi}(v) = \mathrm{diag}(u, u^*)$. By a direct computation one then gets

$$\tilde{\psi}\left(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*\right) = \mathrm{diag}(\mathbf{1}_n, 0_n)$$

which implies in particular that $\tilde{\psi}\left(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*\right)$ is a scalar element of $\widetilde{\mathcal{Q}}$. One then infers from Lemma 4.3.1 that $\tilde{\varphi}$ is injective, and that there exists an element $p \in M_{2n}(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(p) = v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*$. Note that since $v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*$ is a projection and $\tilde{\varphi}$ is injective, then $p$ is a projection as well. Finally, since

$$\tilde{\psi}\big(\tilde{\varphi}(p)\big) = \tilde{\psi}\left(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*\right) = \mathrm{diag}(\mathbf{1}_n, 0_n),$$

one infers that $s(p) = \mathrm{diag}(\mathbf{1}_n, 0_n)$.

(ii) The same arguments as above show that $s(q) = \mathrm{diag}(\mathbf{1}_n, 0_n)$. Note also that $\tilde{\psi}(wv^*) = \mathbf{1}_{2n}$. Again by Lemma 4.3.1 one infers that there exists $z \in M_{2n}(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(z) = wv^*$. Note also that because of the injectivity of $\tilde{\varphi}$, $z$ is necessarily unitary. Finally, since

$$\tilde{\varphi}(zpz^*) = wv^*\tilde{\varphi}(p)vw^* = w\,\mathrm{diag}(\mathbf{1}_n, 0_n)w^* = \tilde{\varphi}(q),$$

one deduces that $q = zpz^*$, which means that $p \sim_u q$ in $\mathcal{P}_{2n}(\widetilde{\mathcal{J}})$, as claimed. □

Based on these results, let us define $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{Q}}) \to K_0(\mathcal{J})$ by $\nu(u) = [p]_0 - [s(p)]_0$ for any $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$, where $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ is the one mentioned in the point $(i)$ of the previous lemma. Note that this map is well-defined because of the point $(ii)$ above. In the following lemma, we gather some additional information on this map $\nu$.

**Lemma 6.1.2.** *The map $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{Q}}) \to K_0(\mathcal{J})$ has the following properties:*

*(i)* $\nu(u_1 \oplus u_2) = \nu(u_1) + \nu(u_2)$ *for any* $u_1, u_2 \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$,

*(ii)* $\nu(\mathbf{1}) = 0$,

*(iii) If* $u_1, u_2 \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ *and* $u_1 \sim_h u_2$, *then* $\nu(u_1) = \nu(u_2)$,

*(iv)* $\nu\big(\tilde{\psi}(u)\big) = 0$ *for any* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

*(v)* $\big[K_0(\varphi)\big]\big(\nu(u)\big) = 0$ *for any* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$.

*Proof.* (i) For $j \in \{1, 2\}$, consider $u_j \in \mathcal{U}_{n_j}(\widetilde{\mathcal{Q}})$ and chose $v_j \in \mathcal{U}_{2n_j}(\widetilde{\mathcal{C}})$ and $p_j \in \mathcal{P}_{2n_j}(\widetilde{\mathcal{J}})$ according to Lemma 6.1.1.$(i)$. In particular, one has

$$\tilde{\psi}(v_j) = \begin{pmatrix} u_j & 0 \\ 0 & u_j^* \end{pmatrix}, \quad \tilde{\varphi}(p_j) = v_j \begin{pmatrix} \mathbf{1}_{n_j} & 0 \\ 0 & 0 \end{pmatrix} v_j^*$$

so that $\nu(u_j) = [p_j]_0 - [s(p_j)]_0$. Let us also introduce the elements $y \in \mathcal{U}_{2(n_1+n_2)}(\mathbb{C})$, $v \in \mathcal{U}_{2(n_1+n_2)}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2(n_1+n_2)}(\widetilde{\mathcal{J}})$ by

$$y = \begin{pmatrix} \mathbf{1}_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n_2} \end{pmatrix}, \quad v = y \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} y^*, \quad p = y \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} y^*.$$

It then follows that

$$\tilde{\psi}(v) = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_1^* & 0 \\ 0 & 0 & 0 & u_2^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_{n_1+n_2} & 0 \\ 0 & 0 \end{pmatrix} v^*,$$

which corresponds to the requirements of Lemma 6.1.1.$(i)$ for $\mathrm{diag}(u_1, u_2)$, and therefore

$$\nu(u_1 \oplus u_2) = [p]_0 - [s(p)]_0 = [p_1 \oplus p_2]_0 - [s(p_1 \oplus p_2)]_0 = \nu(u_1) + \nu(u_2)$$

because $p \sim_u p_1 \oplus p_2$.

(iii) Given $u_1 \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ choose $v_1 \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p_1 \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ such that

$$\tilde{\psi}(v_1) = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix}, \quad \tilde{\varphi}(p_1) = v_1 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_1^*.$$

Then $\nu(u_1) = [p_1]_0 - [s(p_1)]_0$. Since $u_1^* u_2 \sim_h \mathbf{1}_n \sim_h u_1 u_2^*$ in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ we can apply Lemma 2.1.7.$(iii)$ and infer that there exist $a, b \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $\tilde{\psi}(a) = u_1^* u_2$ and $\tilde{\psi}(b) = u_1 u_2^*$. By setting then $v_2 := v_1 \,\mathrm{diag}(a, b) \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ we obtain that

$$\tilde{\psi}(v_2) = \begin{pmatrix} u_2 & 0 \\ 0 & u_2^* \end{pmatrix}, \quad v_2 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_2^* = v_1 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_1^* = \tilde{\varphi}(p_1).$$

Thus, one can choose $p_2 = p_1$ and its satisfies $\tilde{\varphi}(p_2) = v_2 \,\mathrm{diag}(\mathbf{1}_n, 0_n)\, v_2^*$. Finally, by the definition of $\nu$ one infers that $\nu(u_2) = [p_1]_0 - [s(p_1)]_0 = \nu(u_1)$.

(iv) For $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ let us set $v = \mathrm{diag}(u, u^*) \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p = \mathrm{diag}(\mathbf{1}_n, 0_n) \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ so that $p = s(p)$. It then follows that

$$\tilde{\psi}(v) = \begin{pmatrix} \tilde{\psi}(u) & 0 \\ 0 & \tilde{\psi}(u^*) \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*,$$

and thus $\nu\big(\tilde{\psi}(u)\big) = [p]_0 - [s(p)]_0 = 0$.

The statement $(ii)$ is then a direct consequence of $(iv)$, and the statement $(v)$ follows from the fact that $\tilde{\varphi}(p)$ is unitarily equivalent to $s\big(\tilde{\varphi}(p)\big)$ in $M_{2n}(\widetilde{\mathcal{C}})$ when $p$ is a projection in $M_{2n}(\widetilde{\mathcal{J}})$ associated with $u$ as in Lemma 6.1.1.$(i)$. $\square$

By the previous lemma, one deduces that the three conditions required in Proposition 5.1.4 are satisfied. Thus it follows from the universal property of $K_1$ that there exists a unique group homomorphism $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ satisfying $\delta_1([u]_1) = \nu(u)$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$.

**Definition 6.1.3.** *The unique group homomorphism $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ which satisfies*

$$\delta_1([u]_1) = \nu(u) \qquad \forall u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$$

*is called* the index map *associated with the short exact sequence* (6.1).

Let us summarize the main properties of the index map in the following statement:

**Proposition 6.1.4** (First standard picture of the index map)**.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*be a short sequence of $C^*$-algebras, let $n$ be a natural number, and consider $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$, $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ which satisfy*

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*.$$

*Then*

$$\delta_1([u]_1) = [p]_0 - [s(p)]_0.$$

*Moreover, one has*

*(i)* $\delta_1 \circ K_1(\psi) = 0$,

*(ii)* $K_0(\varphi) \circ \delta_1 = 0$.

**Proposition 6.1.5** (Naturality of the index map)**.** *Let*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0
\end{tikzcd}$$

*be a commutative diagram with two short exact sequences of $C^*$-algebras, and with $\alpha, \beta, \gamma$ three $*$-homomorphisms. Let $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ and $\delta_1' : K_1(\mathcal{Q}') \to K_0(\mathcal{J}')$ be the index map associated with the short exact sequences. Then the following diagram is commutative:*

$$\begin{tikzcd}
K_1(\mathcal{Q}) \arrow[r, "\delta_1"] \arrow[d, "K_1(\beta)"] & K_0(\mathcal{J}) \arrow[d, "K_0(\gamma)"] \\
K_1(\mathcal{Q}') \arrow[r, "\delta_1'"] & K_0(\mathcal{J}') \ .
\end{tikzcd}$$

*Proof.* Let $g$ be an element in $K_1(\mathcal{Q})$, and let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ with $g = [u]_1$. By Lemma 6.1.1.$(i)$ there exists $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ such that

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*.$$

Set $v' := \tilde{\alpha}(v)$ in $\mathcal{U}_{2n}(\widetilde{\mathcal{C}'})$ and $p' := \tilde{\gamma}(p)$ in $\mathcal{P}_{2n}(\widetilde{\mathcal{J}'})$. Then one has

$$\begin{aligned} \tilde{\psi}'(v') &= (\widetilde{\psi' \circ \alpha})(v) = (\widetilde{\beta \circ \psi})(v) = \tilde{\beta}\big(\tilde{\psi}(v)\big) = \begin{pmatrix} \tilde{\beta}(u) & 0 \\ 0 & \tilde{\beta}(u)^* \end{pmatrix}, \\ \tilde{\varphi}'(p') &= (\widetilde{\varphi' \circ \gamma})(p) = (\widetilde{\alpha \circ \varphi})(p) = \tilde{\alpha}(v) \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \tilde{\alpha}(v)^* = v' \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} (v')^* , \end{aligned}$$

which corresponds to the requirements of Lemma 6.1.1.$(i)$ for $\tilde{\beta}(u)$, Then, from the definition of the index map one has

$$\begin{aligned} \big(\delta_1' \circ K_1(\beta)\big)(g) &= \delta_1'\big([\tilde{\beta}(u)]_1\big) = [p']_0 - [s(p')]_0 \\ &= [\tilde{\gamma}(p)]_0 - \big[s\big(\tilde{\gamma}(p)\big)\big]_0 = K_0(\gamma)\big([p]_0 - [s(p)]_0\big) \\ &= K_0(\gamma)\big(\delta_1([u]_1)\big) = \big(K_0(\gamma) \circ \delta_1\big)(g). \end{aligned}$$

This shows that $\delta_1' \circ K_1(\beta) = K_0(\gamma) \circ \delta_1$. □

## 6.2 The index map and partial isometries

In this section we provide another picture of the index map, which is more intuitive and more useful in applications. The key point in the construction is the following lemma.

**Lemma 6.2.1.** *Let $\psi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism between $C^*$-algebras, and suppose that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and $\psi$ is unit preserving). Then for each unitary element $u \in \mathcal{Q}$ there exists a partial isometry $v \in M_2(\mathcal{C})$ such that*

$$\psi(v) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.3}$$

*Proof.* For any $u \in \mathcal{U}(\mathcal{Q})$, there exists by Proposition 2.3.1.$(i)$ an element $c \in \mathcal{C}$ such that $\psi(c) = u$ and $\|c\| = \|u\| = 1$. We then set

$$v = \begin{pmatrix} c & 0 \\ (\mathbf{1} - c^*c)^{1/2} & 0 \end{pmatrix}$$

and check that $v^*v = \mathrm{diag}(\mathbf{1}, 0)$. By taking Exercise 2.2.3 into account, one infers that $vv^*$ is a projection as well, and that $v$ is a partial isometry. From the equalities $\psi(c) = u$ and $\psi\big((\mathbf{1} - c^*c)^{1/2}\big) = (\mathbf{1} - u^*u)^{1/2} = 0$, one deduces that (6.3) holds. □

**Proposition 6.2.2** (Second standard picture of the index map)**.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*be a short sequence of $C^*$-algebras. Let $n \leq m$ be a natural numbers, let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ and let $v$ be a partial isometry in $M_m(\widetilde{\mathcal{C}})$ with*

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & 0_{m-n} \end{pmatrix}. \tag{6.4}$$

*Then $\mathbf{1}_m - v^*v = \tilde{\varphi}(p)$ and $\mathbf{1}_m - vv^* = \tilde{\varphi}(q)$ for some projections $p, q$ in $\mathcal{P}_m(\widetilde{\mathcal{J}})$, and the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ is given by*

$$\delta_1([u]_1) = [p]_0 - [q]_0. \tag{6.5}$$

Before providing the proof observe that Lemma 6.2.1 ensures the existence of a partial isometry $v$ satisfying (6.4). It is then a consequence of the above proposition that the r.h.s. of (6.5) does not depend on the choice of $v$.

*Proof.* Let us set $e = \mathbf{1}_m - v^*v$ and $f = \mathbf{1}_m - vv^*$ in $\mathcal{P}_m(\widetilde{\mathcal{C}})$. Then one has $\tilde{\psi}(e) = \tilde{\psi}(f) = \mathrm{diag}(0_n, \mathbf{1}_{m-n})$. Because $\tilde{\psi}(e)$ and $\tilde{\psi}(f)$ are scalar matrices, it follows from Lemma 4.3.1.(*ii*) that there are projections $p, q \in \mathcal{P}_m(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(p) = e$ and $\tilde{\varphi}(q) = f$, and $s(p) = s(q) = \mathrm{diag}(0_n, \mathbf{1}_{m-n})$. Let us then set

$$w := \begin{pmatrix} v & f \\ e & v^* \end{pmatrix}, \quad r := \begin{pmatrix} \mathbf{1}_m - q & 0 \\ 0 & p \end{pmatrix}, \quad z := \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m-n} \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_{m-n} & 0 & 0 \end{pmatrix}.$$

Then $r$ is a projection in $M_{2m}(\widetilde{\mathcal{J}})$, $w$ is a unitary element of $M_{2m}(\widetilde{\mathcal{C}})$ and $z$ is a self-adjoint unitary matrix in $M_{2m}(\mathbb{C})$. In addition, $zw \in \mathcal{U}_{2m}(\widetilde{\mathcal{C}})$, and one has

$$\tilde{\psi}(zw) = z \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m-n} \\ 0 & 0 & u^* & 0 \\ 0 & \mathbf{1}_{m-n} & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix},$$

where $u_1 = \mathrm{diag}(u, \mathbf{1}_{m-n})$ in $\mathcal{U}_m(\widetilde{\mathcal{Q}})$. One also observes that

$$zw \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & 0 \end{pmatrix} w^*z^* = z \begin{pmatrix} vv^* & ve \\ ev^* & e \end{pmatrix} z^* = z \begin{pmatrix} \mathbf{1}_m - f & 0 \\ 0 & e \end{pmatrix} z^* = \tilde{\varphi}(zrz^*).$$

Since $zrz \in \mathcal{P}_{2m}(\widetilde{\mathcal{J}})$, it finally follows from the definition of the index map that

$$\begin{aligned} \delta_1([u]_1) = \delta_1([u_1]_1) &= [zrz^*]_0 - [s(zrz^*)]_0 = [r]_0 - [s(r)]_0 \\ &= [\mathbf{1}_m - q]_0 + [p]_0 - [\mathbf{1}_n]_0 - [\mathbf{1}_{m-n}]_0 = [p]_0 - [q]_0, \end{aligned}$$

as desired. □

Note that if $\mathcal{J}$ is an ideal in $\mathcal{C}$ and if $\varphi$ is the inclusion map, then (6.5) can be rephrased as

$$\delta_1([u]_1) = [\mathbf{1}_m - v^*v]_0 - [\mathbf{1}_m - vv^*]_0, \tag{6.6}$$

where $m, n$ are integers with $m \geq n$, $u$ belongs to $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ and $v$ is a partial isometry in $M_m(\widetilde{\mathcal{C}})$ that lifts $\mathrm{diag}(u, 0_{m-n})$.

Similarly, if $\mathcal{C}$ and $\mathcal{Q}$ are unital $C^*$-algebras, it would be convenient to have a direct expression for the index map. The following statement deals with such a situation for both pictures of the index map.

**Proposition 6.2.3.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*be a short sequence of $C^*$-algebras, and suppose that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and $\psi$ is unit preserving). Let $\bar{\varphi} : \widetilde{\mathcal{J}} \to \mathcal{C}$ be the $*$-homomorphism defined by $\bar{\varphi}(a + \alpha \mathbf{1}_{\widetilde{\mathcal{C}}}) = \varphi(a) + \alpha \mathbf{1}_{\mathcal{C}}$ for any $a \in \mathcal{J}$ and $\alpha \in \mathbb{C}$. Let also $u \in \mathcal{U}_n(\mathcal{Q})$.*

*(i) If $v \in \mathcal{U}_{2n}(\mathcal{C})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ are such that*

$$\bar{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad \psi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix},$$

*then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$,*

*(ii) Let $m \geq n$ be integers and let $v$ be partial isometry in $M_m(\mathcal{C})$ with $\psi(v) = \mathrm{diag}(u, 0_{m-n})$, then $\mathbf{1}_m - v^*v = \bar{\varphi}(p)$ and $\mathbf{1}_m - vv^* = \bar{\varphi}(q)$ for some $p, q$ in $\mathcal{P}_m(\widetilde{\mathcal{J}})$, and $\delta_1([u]_1) = [p]_0 - [q]_0$.*

We refer to [RLL00, Prop. 9.2.3] for the proof of the above statement. Let us provide one more version of the previous results when $u$ in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ or $u$ in $\mathcal{U}_n(\mathcal{Q})$ lifts to a partial isometry in $M_n(\widetilde{\mathcal{Q}})$ or in $M_n(\mathcal{Q})$, respectively, and where for further simplification we assume that $\mathcal{J}$ is an ideal of $\mathcal{C}$.

**Proposition 6.2.4.** *Let*

$$0 \longrightarrow \mathcal{J} \xhookrightarrow{\iota} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*be a short sequence of $C^*$-algebras, where $\mathcal{J}$ is an ideal in $\mathcal{C}$ and $\iota$ is the inclusion map.*

*(i) Let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ and let $v \in M_n(\widetilde{\mathcal{C}})$ be a partial isometry such that $\tilde{\psi}(v) = u$. Then $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ are projections in $M_n(\mathcal{J})$, and*

$$\delta_1([u]_1) = [\mathbf{1}_n - v^*v]_0 - [\mathbf{1}_n - vv^*]_0. \tag{6.7}$$

*(ii) Assume that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and $\psi$ is unit preserving), and let $u \in \mathcal{U}_n(\mathcal{Q})$ which has a lift to a partial isometry $v \in M_n(\mathcal{C})$. Then $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ are projections in $M_n(\mathcal{J})$, and*

$$\delta_1([u]_1) = [\mathbf{1}_n - v^*v]_0 - [\mathbf{1}_n - vv^*]_0.$$

*Proof.* (i) Since

$$\tilde{\psi}(\mathbf{1}_n - v^*v) = \mathbf{1}_n - u^*u = 0, \qquad \tilde{\psi}(\mathbf{1}_n - vv^*) = \mathbf{1}_n - uu^* = 0, \tag{6.8}$$

we see that $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ belong to $M_n(\mathcal{J})$, and these two elements are projections because $v$ is a partial isometry. The identity (6.7) follows from Proposition 6.2.2 together with (6.6).

(ii) Here $\psi(\mathbf{1}_n - v^*v) = \mathbf{1}_n - u^*u = 0$ and $\psi(\mathbf{1}_n - vv^*) = \mathbf{1}_n - uu^* = 0$, showing that $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ belong to $M_n(\mathcal{J})$. The statement can then be inferred from Proposition 6.2.3.(*ii*). □

## 6.3 An exact sequence of $K$-groups

In this section, we show that the exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

induces the exact sequence (6.2) at the level of $K$-groups, with $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ the index map introduced in the previous sections. For that purpose and without loss of generality, we shall assume that $\mathcal{J}$ is an ideal in $\mathcal{C}$ and that the map $\varphi$ is the inclusion map. In this case, $M_n(\widetilde{\mathcal{J}})$ is a unital $C^*$-subalgebra of $M_n(\widetilde{\mathcal{C}})$ for each $n \in \mathbb{N}^*$.

**Lemma 6.3.1.** *The kernel of the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ is contained in the image of the map $K_1(\psi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$.*

*Proof.* Let $g \in K_1(\mathcal{Q})$ such that $\delta_1(g) = 0$, and let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ such that $g = [u]_1$. By Lemma 6.2.1 there exists a partial isometry $w_1 \in M_{2n}(\widetilde{\mathcal{C}})$ such that

$$\tilde{\psi}(w_1) = \begin{pmatrix} u & 0 \\ 0 & 0_n \end{pmatrix}.$$

Then, by Proposition 6.2.2 one infers that the following equalities hold in $K_0(\mathcal{J})$:

$$0 = \delta_1(g) = \delta_1([u]_1) = [\mathbf{1}_{2n} - w_1^*w_1]_0 - [\mathbf{1}_{2n} - w_1w_1^*]_0 \ .$$

By Proposition 3.2.4, one infers that there exists $k \in \mathbb{N}$ and a partial isometry $w_2 \in M_{2n+k}(\widetilde{\mathcal{J}})$ such that

$$(\mathbf{1}_{2n} - w_1^*w_1) \oplus \mathbf{1}_k = w_2^*w_2, \quad (\mathbf{1}_{2n} - w_1w_1^*) \oplus \mathbf{1}_k = w_2w_2^*.$$

Computing the image of these elements through $\tilde{\psi}$ one gets

$$\tilde{\psi}(w_2^*w_2) = \begin{pmatrix} 0_n & 0 \\ 0 & \mathbf{1}_{n+k} \end{pmatrix} = \tilde{\psi}(w_2w_2^*).$$

In addition, since $w_2 \in M_{2n+k}(\widetilde{\mathcal{J}})$ one deduces from Lemma 4.3.1 that $\tilde{\psi}(w_2)$ is a scalar matrix. As a consequence, one has $\tilde{\psi}(w_2) = \mathrm{diag}(0_n, z)$ for some scalar and unitary

matrix $z \in M_{n+k}(\widetilde{\mathcal{Q}})$. Since $\mathcal{U}_{n+k}(\mathbb{C})$ is connected, *cf.* Corollary 2.1.3, one finally deduces that $z$ is homotopic to $\mathbf{1}_{n+k}$ in $\mathcal{U}_{n+k}(\widetilde{\mathcal{Q}})$.

Let us now set $v := \operatorname{diag}(w_1, 0_k) + w_2$. One can observe that $v \in \mathcal{U}_{2n+k}(\widetilde{\mathcal{C}})$, and that

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & 0_{n+k} \end{pmatrix} + \begin{pmatrix} 0_n & 0 \\ 0 & z \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & \mathbf{1}_{n+k} \end{pmatrix} \quad \text{in } \mathcal{U}_{2n+k}(\widetilde{\mathcal{Q}}).$$

This proves that

$$g = [u]_1 = [\tilde{\psi}(v)]_1 = K_1(\psi)([v]_1),$$

as desired. □

**Lemma 6.3.2.** *The kernel of the map $K_0(\varphi) : K_0(\mathcal{J}) \to K_0(\mathcal{C})$ is contained in the image of the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$.*

*Proof.* Let $g \in K_0(\mathcal{J})$ with $g \in \mathsf{Ker}\big(K_0(\varphi)\big)$. By Lemma 4.2.2, there exist $n \in \mathbb{N}$, $p \in \mathcal{P}_n(\widetilde{\mathcal{J}})$ and $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $wpw^* = s(p)$.

The element $u_0 := \tilde{\psi}\big(w(\mathbf{1}_n - p)\big)$ is a partial isometry in $M_n(\widetilde{\mathcal{Q}})$ and

$$\mathbf{1}_n - u_0^* u_0 = \tilde{\psi}(p) = \tilde{\psi}\big(s(p)\big) = \mathbf{1}_n - u_0 u_0^*,$$

where Lemma 4.3.1 has been used for the second equality. It follows that $u_0$ is a partial isometry and is normal, and that

$$u := u_0 + (\mathbf{1}_n - u_0^* u_0)$$

is a unitary element in $M_n(\widetilde{\mathcal{Q}})$. In order to lift $\operatorname{diag}(u, 0_n)$ to a suitable partial isometry $v$ in $M_{2n}(\widetilde{\mathcal{C}})$, let us first observe that $v_1 := \operatorname{diag}\big(w(\mathbf{1}_n - p), s(p)\big)$ in $M_{2n}(\widetilde{\mathcal{C}})$ satisfies $\tilde{\psi}(v_1) = \operatorname{diag}\big(u_0, s(p)\big)$. Let $z \in M_{2n}(\mathbb{C})$ be the self-adjoint unitary matrix given by

$$z := \begin{pmatrix} \mathbf{1}_n - s(p) & s(p) \\ s(p) & \mathbf{1}_n - s(p) \end{pmatrix},$$

and set $v := z v_1 z^*$. Then one has

$$\tilde{\psi}(v) = z\tilde{\psi}(v_1) z^* = z \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix} z^* = \begin{pmatrix} u & 0 \\ 0 & 0_n \end{pmatrix}.$$

It finally follows from Proposition 6.2.2 that

$$\begin{aligned} \delta_1([u]_1) &= [\mathbf{1}_{2n} - v^* v]_0 - [\mathbf{1}_{2n} - v v^*]_0 = [\mathbf{1}_{2n} - v_1^* v_1]_0 - [\mathbf{1}_{2n} - v_1 v_1^*]_0 \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & \mathbf{1}_n - s(p) \end{pmatrix}\right]_0 - \left[\begin{pmatrix} s(p) & 0 \\ 0 & \mathbf{1}_n - s(p) \end{pmatrix}\right]_0 \\ &= [p]_0 - [s(p)]_0 \\ &= g \end{aligned}$$

in $K_0(\mathcal{J})$, and this proves the statement. □

By combining the contents of Propositions 4.3.2, 5.2.3, and 6.1.4, together with Lemmas 6.3.1 and 6.3.2 one gets the following result:

**Proposition 6.3.3.** *Every short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*gives rise to the following exact sequence of Abelian groups:*

$$\begin{tikzcd}
K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\
K_0(\mathcal{Q}) & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"] \; .
\end{tikzcd}$$

**Extension 6.3.4.** *Study the classical situation of Fredholm operators and Fredholm index, as presented for example in [RLL00, Sec. 9.4].*

# Chapter 7

# Higher $K$-functors, Bott periodicity

In this chapter, we first show that $K_1(\mathcal{C})$ is isomorphic to $K_0\big(S(\mathcal{C})\big)$, where $S(\mathcal{C})$ is the suspension of a $C^*$-algebra $\mathcal{C}$ defined in (4.3). Higher $K$-groups are then defined iteratively, and various exact sequences are considered. The Bott map is constructed and Bott periodicity is stated. However, its full proof is not provided.

## 7.1 The isomorphism between $K_1(\mathcal{C})$ and $K_0\big(S(\mathcal{C})\big)$

Let us first recall that the suspension of an arbitrary $C^*$-algebra $\mathcal{C}$ is defined by

$$S(\mathcal{C}) := \big\{f \in C([0,1];\mathcal{C}) \mid f(0) = f(1) = 0\big\}$$

and observe that this $C^*$-algebra is equal to $C_0\big((0,1);\mathcal{C}\big)$. Clearly, the norm on $S(\mathcal{C})$ is defined by $\|f\| := \sup_{t\in[0,1]} \|f(t)\|_{\mathcal{C}}$, and $f^*(t) := f(t)^*$. With any $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$ one can associate a $*$-homomorphism $S(\varphi) : S(\mathcal{C}) \to S(\mathcal{Q})$ by $[S(\varphi)(f)](t) := \varphi\big(f(t)\big)$ for any $f \in S(\mathcal{C})$ and $t \in [0,1]$. In this way, $S$ defines a functor from the category of $C^*$-algebras to itself, with $S(\{0\}) = \{0\}$ and $S(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{S(\mathcal{C})\to S(\mathcal{Q})}$.

The following lemma is a classical statement about density. Its proof is left to the reader, see also [RLL00, Lemma 10.1.1].

**Lemma 7.1.1.** *Let $\Omega$ be a locally compact Hausdorff space and let $\mathcal{C}$ be a $C^*$-algebra. For any $f \in C_0(\Omega)$ and any $a \in \mathcal{C}$ one writes $fa$ for the element of $C_0(\Omega;\mathcal{C})$ defined by $[fa](x) = f(x)a$ for any $x \in \Omega$. Then the set*

$$\operatorname{span}\{fa \mid f \in C_0(\Omega), a \in \mathcal{C}\}$$

*is dense in $C_0(\Omega;\mathcal{C})$.*

We can now show the main result about the functor $S$:

**Lemma 7.1.2** (Exactness of $S$)**.** *The functor $S$ is exact.*

*Proof.* Given the short exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

one has to show that

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0$$

is also a short exact sequence. In fact, the only non-trivial part is to show that $S(\psi)$ is surjective. However, this easily follows from the density of $\mathrm{span}\{fb \mid f \in C_0(\Omega), b \in \mathcal{Q}\}$ in $S(\mathcal{Q})$ and from the fact that any element of this dense set belongs to the range of $S(\mathcal{C})$ by $S(\psi)$, since $S(\psi)(fa) = f\psi(a)$ for any $a \in \mathcal{C}$ and any $f \in C_0\big((0,1)\big)$. □

**Theorem 7.1.3.** *For any $C^*$-algebra $\mathcal{C}$ there exists an isomorphism*

$$\theta_{\mathcal{C}} : K_1(\mathcal{C}) \to K_0\big(S(\mathcal{C})\big)$$

*satisfying the following property: If $\varphi$ is a $*$-homomorphism between two $C^*$-algebras $\mathcal{C}$ and $\mathcal{Q}$ then the following diagram is commutative:*

$$\begin{tikzcd} K_1(\mathcal{C}) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_{\mathcal{C}}"] & K_1(\mathcal{Q}) \arrow[d, "\theta_{\mathcal{Q}}"] \\ K_0\big(S(\mathcal{C})\big) \arrow[r, "K_0(S(\varphi))"] & K_0\big(S(\mathcal{Q})\big) \end{tikzcd} \tag{7.1}$$

*Proof.* Let us first consider the short exact sequence

$$0 \longrightarrow S(\mathcal{C}) \xhookrightarrow{\iota} C(\mathcal{C}) \xrightarrow{\pi} \mathcal{C} \longrightarrow 0, \tag{7.2}$$

where $C(\mathcal{C})$ denotes the cone of $\mathcal{C}$. Since $C(\mathcal{C})$ is homotopy equivalent to $\{0\}$, as shown at the end of Section 4.1, it follows that $K_0\big(C(\mathcal{C})\big) = K_1\big(C(\mathcal{C})\big) = \{0\}$. By applying then the exact sequence of Abelian groups obtained in Proposition 6.3.3 to the above short exact sequence of $C^*$-algebras one infers that the map $\delta_1 : K_1(\mathcal{C}) \to K_0\big(S(\mathcal{C})\big)$ is an isomorphism. One can thus set $\theta_{\mathcal{C}} = \delta_1$.

Observe now that every $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S(\mathcal{C}) \arrow[r] \arrow[d, "S(\varphi)"] & C(\mathcal{C}) \arrow[r] \arrow[d, "C(\varphi)"] & \mathcal{C} \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & S(\mathcal{Q}) \arrow[r] & C(\mathcal{Q}) \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

where $[C(\varphi)(f)](t) := \varphi\big(f(t)\big)$ for any $f \in C(\mathcal{C})$ and $t \in [0,1]$. By applying then the naturality of the index map, see Proposition 6.1.5, one directly gets the commutative diagram (7.1). □

For later use, let us provide a more concrete description of the isomorphism $\theta_{\mathcal{C}}$. For that purpose, let $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $s(u) = \mathbf{1}_n$ be given. Let $v \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ be such that $v(0) = \mathbf{1}_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$, and $s\big(v(t)\big) = \mathbf{1}_{2n}$ for any $t \in [0,1]$, and set $p := v\,\mathrm{diag}(\mathbf{1}_n, 0)v^*$. Then $p \in \mathcal{P}_{2n}\big(\widetilde{S(\mathcal{C})}\big)$, $s(p) = \mathrm{diag}(\mathbf{1}_n, 0)$ and

$$\theta_{\mathcal{C}}([u]_1) := [p]_0 - [s(p)]_0 \ .$$

For the justification of this formula observe first that any $g \in K_1(\mathcal{C})$ can be represented by an element $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $s(u) = \mathbf{1}_n$. Indeed, for any $g \in K_1(\mathcal{C})$ there exists $n \in \mathbb{N}$ and $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [w]_1$. Then one can set $u := ws(w)^*$ and check that $s(u) = \mathbf{1}_n$ and $g = [u]_1$. Note that the latter equality holds since $s(w)^* \sim_h \mathbf{1}_n$, and this follows from Corollary 2.1.3 about the property that the unitary group in $M_n(\mathbb{C})$ is connected.

Now, for each $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $s(u) = \mathbf{1}_n$ we can find $v \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ with $v(0) = \mathbf{1}_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$ and $s\big(v(t)\big) = \mathbf{1}_{2n}$ for every $t \in [0,1]$. Indeed, by Whitehead's Lemma (Lemma 2.1.4) one can find $z \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ with $z(0) = \mathbf{1}_{2n}$ and $z(1) = \mathrm{diag}(u, u^*)$. The element $v$ is then defined by $v(t) := s\big(z(t)\big)^* z(t)$ and has the desired properties.

Let us finally observe that an element $f \in C\big([0,1]; M_{2n}(\widetilde{\mathcal{C}})\big)$ belongs to $M_{2n}\big(\widetilde{C(\mathcal{C})}\big)$ if and only if $s\big(f(t)\big) = f(0)$ for each $t \in [0,1]$, while $f$ belongs to $M_{2n}\big(\widetilde{S(\mathcal{C})}\big)$ if and only if $s\big(f(t)\big) = f(0) = f(1)$ for each $t \in [0,1]$. Note also that if $\pi$ is defined as in (7.2), then $\tilde{\pi}(f) = f(1)$ for any $f \in M_{2n}\big(\widetilde{C(\mathcal{C})}\big)$. With these identifications, it follows that $v \in \mathcal{U}_{2n}\big(\widetilde{C(\mathcal{C})}\big)$, and

$$\tilde{\pi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad p = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v \ \in \mathcal{P}_{2n}\big(\widetilde{S(\mathcal{C})}\big).$$

By the definition of the index map, one infers that

$$\theta_{\mathcal{C}}([u]_1) = \delta_1([u]_1) = [p]_0 - [s(p)]_0,$$

as already mentioned.

## 7.2 The long exact sequence in $K$-theory

In this section we define the higher functor $K_n$ for every integer $n \geq 2$. Part of the construction should be considered as a preliminary step for the six-term exact sequence which will be obtained later on.

**Definition 7.2.1.** *For each integer $n \geq 2$ one defines iteratively the functor $K_n$ from the category of $C^*$-algebras to the category of Abelian groups by*

$$K_n := K_{n-1} \circ S$$

*where the suspension $S$ is seen as a functor from the category of $C^*$-algebras into itself.*

More specifically, for any $n \geq 2$ and for any $C^*$-algebra $\mathcal{C}$ one sets

$$K_n(\mathcal{C}) := K_{n-1}\big(S(\mathcal{C})\big)$$

and for each $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between $C^*$-algebras one also sets

$$K_n(\varphi) := K_{n-1}\big(S(\varphi)\big).$$

Now, let us denote by $S^n(\mathcal{C})$ the $n$-th iterated suspension of the $C^*$-algebra $\mathcal{C}$. It is inductively defined by $S^n(\mathcal{C}) := S\big(S^{n-1}(\mathcal{C})\big)$. Similarly, if $\mathcal{Q}$ is another $C^*$-algebra and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, then one gets a $*$-homomorphism $S^n(\varphi) : S^n(\mathcal{C}) \to S^n(\mathcal{Q})$. This $*$-homomorphism is defined by induction by the relation $S^n(\varphi) = S\big(S^{n-1}(\varphi)\big)$. The higher $K$-groups are then given by

$$K_n(\mathcal{C}) = K_1\big(S^{n-1}(\mathcal{C})\big) \cong K_0\big(S^n(\mathcal{C})\big), \tag{7.3}$$

and

$$K_n(\varphi) = K_1\big(S^{n-1}(\varphi)\big). \tag{7.4}$$

We shall also apply the convention that $S^0(\mathcal{C}) = \mathcal{C}$ and $S^0(\varphi) = \varphi$.

**Proposition 7.2.2.** *For each integer $n \geq 2$, $K_n$ is a half exact functor from the category of $C^*$-algebras to the category of Abelian groups.*

*Proof.* As already mentioned, the suspension $S$ is an exact functor from the category of $C^*$-algebras to itself, see Lemma 7.1.2. On the other hand, $K_1$ is a half exact functor, as shown in Proposition 5.2.3. Since the composition of two functors is again a functor, we obtain by formulas (7.3) and (7.4) that $K_n$ is a functor for each $n \geq 2$. The half exactness of $K_n$ easily follows from the mentioned properties of $S$ and of $K_1$. □

For the short exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

let us now define the higher index maps. For that purpose and for $n \geq 1$ one defined inductively the index maps $\delta_{n+1} : K_{n+1}(\mathcal{Q}) \to K_n(\mathcal{J})$ as follows. By the exactness of $S$, the sequence

$$0 \longrightarrow S^n(\mathcal{J}) \xrightarrow{S^n(\varphi)} S^n(\mathcal{C}) \xrightarrow{S^n(\psi)} S^n(\mathcal{Q}) \longrightarrow 0 \tag{7.5}$$

is exact, and by Theorem 7.1.3 we have an isomorphism

$$\theta_{S^{n-1}(\mathcal{J})} : K_n(\mathcal{J}) = K_1\big(S^{n-1}(\mathcal{J})\big) \to K_0\big(S^n(\mathcal{J})\big).$$

As a consequence, there exists one and only one group homomorphism $\delta_{n+1}$ making the diagram

$$\begin{tikzcd}
K_{n+1}(\mathcal{Q}) \arrow[r, "\delta_{n+1}"] \arrow[d, "\mathrm{id}"] & K_n(\mathcal{J}) \\
K_1\big(S^n(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] & K_0\big(S^n(\mathcal{J})\big) \arrow[u, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"']
\end{tikzcd} \tag{7.6}$$

commutative, where $\bar{\delta}_1$ is the index map associated with the short exact sequence (7.5).

Note that the index maps $\delta_1, \delta_2, \ldots$ are natural in the following sense: Given a commutative diagram of $C^*$-algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0 \end{tikzcd} \tag{7.7}$$

with $*$-homomorphisms $\alpha, \beta, \gamma$, then the diagram

$$\begin{tikzcd} K_{n+1}(\mathcal{Q}) \arrow[r, "\delta_{n+1}"] \arrow[d, "K_{n+1}(\beta)"'] & K_n(\mathcal{J}) \arrow[d, "K_n(\gamma)"] \\ K_{n+1}(\mathcal{Q}') \arrow[r, "\delta'_{n+1}"] & K_n(\mathcal{J}') \ . \end{tikzcd} \tag{7.8}$$

is commutative. To see this, let us apply the exact functor $S^n$ to the diagram (7.7), let $\bar{\delta}_1$ and $\bar{\delta}'_1$ be the index maps of the two resulting short exact sequences, and consider the diagram

$$\begin{tikzcd} K_{n+1}(\mathcal{Q}) \arrow[r, "\mathrm{id}"] \arrow[d, "K_{n+1}(\beta)"'] & K_1\big(S^n(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] \arrow[d, "K_1(S^n(\beta))"'] & K_0\big(S^n(\mathcal{J})\big) \arrow[r, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"] \arrow[d, "K_0(S^n(\gamma))"] & K_n(\mathcal{J}) \arrow[d, "K_n(\gamma)"] \\ K_{n+1}(\mathcal{Q}') \arrow[r, "\mathrm{id}"] & K_1\big(S^n(\mathcal{Q}')\big) \arrow[r, "\bar{\delta}'_1"] & K_0\big(S^n(\mathcal{J}')\big) \arrow[r, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"] & K_n(\mathcal{J}') \ . \end{tikzcd} \tag{7.9}$$

The center square of this diagram commutes by naturality of the index map $\delta_1$, see Proposition 6.1.5, and the right-hand square commutes by naturality of $\theta$, as obtained in Theorem 7.1.3. Hence, (7.9) is a commutative diagram. Since $\delta_{n+1}$ corresponds to the composition of the three horizontal homomorphisms, this implies that (7.8) is commutative.

**Proposition 7.2.3** (The long exact sequence in $K$-theory)**.** *Every short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*induces an exact sequence of $K$-groups:*

$$\cdots \xrightarrow{K_{n+1}(\psi)} K_{n+1}(\mathcal{Q}) \xrightarrow{\delta_{n+1}} K_n(\mathcal{J}) \xrightarrow{K_n(\varphi)} K_n(\mathcal{C}) \xrightarrow{K_n(\psi)} K_n(\mathcal{Q}) \xrightarrow{\delta_n} K_{n-1}(\mathcal{J}) \xrightarrow{K_{n-1}(\varphi)} \cdots$$
$$\cdots \xrightarrow{\delta_1} K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \xrightarrow{K_0(\psi)} K_0(\mathcal{Q}),$$

*where $\delta_1$ is the index map and $\delta_n$ its higher analogues for $n \geq 2$.*

*Proof.* Let $\{\bar{\delta}_n\}_{n=1}^{\infty}$ be the index maps associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0.$$

It follows directly from the definition of the index maps and Theorem 7.1.3 that the diagrams

$$\begin{tikzcd}
K_2(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_2(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] \arrow[d, "\mathrm{id}"] & K_1(\mathcal{J}) \arrow[r] \arrow[d, "\theta_{\mathcal{J}}"] & K_1(\mathcal{C}) \arrow[r] \arrow[d, "\theta_{\mathcal{C}}"] & K_1(\mathcal{Q}) \arrow[d, "\theta_{\mathcal{Q}}"] \\
K_1\big(S(\mathcal{J})\big) \arrow[r] & K_1\big(S(\mathcal{C})\big) \arrow[r] & K_1\big(S(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"'] & K_0\big(S(\mathcal{J})\big) \arrow[r] & K_0\big(S(\mathcal{C})\big) \arrow[r] & K_0\big(S(\mathcal{Q})\big)
\end{tikzcd}$$

and, for $n \geq 3$

$$\begin{tikzcd}
K_n(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_n(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_n(\mathcal{Q}) \arrow[r, "\delta_n"] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{Q}) \arrow[d, "\mathrm{id}"] \\
K_{n-1}(S(\mathcal{J})) \arrow[r] & K_{n-1}(S(\mathcal{C})) \arrow[r] & K_{n-1}(S(\mathcal{Q})) \arrow[r, "\delta_{n-1}"'] & K_{n-2}(S(\mathcal{J})) \arrow[r] & K_{n-2}(S(\mathcal{C})) \arrow[r] & K_{n-2}(S(\mathcal{Q}))
\end{tikzcd}$$

are commutative. The lower row in the first diagram is exact by Proposition 6.3.3, and for both diagrams the exactness of the lower row implies the exactness of the upper row. Exactness of the long exact sequence is then established by induction. □

**Example 7.2.4.** *The suspension $S(\mathcal{C}) = C_0\big((0,1);\mathcal{C}\big)$ of a $C^*$-algebra $\mathcal{C}$ is isomorphic to $C_0(\mathbb{R};\mathcal{C})$ since $\mathbb{R}$ is homeomorphic to $(0,1)$. Note also that $C_0\big(X;C_0(Y)\big)$ is isomorphic to $C_0(X \times Y)$ for any pair of locally compact Hausdorff spaces $X$ and $Y$. As a consequence, $S^n(\mathbb{C})$ is isomorphic to $C_0(\mathbb{R}^n)$, from which one infers that*

$$K_n(\mathbb{C}) \cong K_0\big(C_0(\mathbb{R}^n)\big), \qquad K_{n+1}(\mathbb{C}) \cong K_1\big(C_0(\mathbb{R}^n)\big)$$

*for any $n \geq 1$.*

## 7.3 The Bott map

From now on, the following picture for $S(\mathcal{C})$ will be used:

$$S(\mathcal{C}) := \big\{f \in C(\mathbb{T};\mathcal{C}) \mid f(1) = 0\big\}$$

with $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. Although this definition does not corresponds to the previous one, the two algebras are clearly isomorphic.

Let us first consider a unital $C^*$-algebra $\mathcal{C}$. For any $n \in \mathbb{N}^*$ and $p \in \mathcal{P}_n(\mathcal{C})$ one defines *the projection loop* $f_p : \mathbb{T} \to \mathcal{U}_n(\mathcal{C})$ by

$$f_p(z) := zp + (\mathbf{1}_n - p), \qquad \forall z \in \mathbb{T}.$$

By identifying $M_n\big(\widetilde{S(\mathcal{C})}\big)$ with the set of elements $f$ of $C(\mathbb{T}; M_n(\mathcal{C}))$ such that $f(1)$ belongs to $M_n(\mathbb{C}\mathbf{1}_{\mathcal{C}})$, by considering $f - f(1)$ which belongs to $M_n\big(S(\mathcal{C})\big)$, we obtain that $f_p$ belongs to $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$. In addition, observe that the maps $p \mapsto f_p$ and $f_p \mapsto p$ are continuous because of the equalities

$$\|f_p - f_q\| = \sup_{z\in\mathbb{T}} \|f_p(z) - f_q(z)\| = 2\|p - q\|.$$

One then easily infers that the following properties hold:

(i) $f_{p\oplus q} = f_p \oplus f_q$ for any projections $p, q \in \mathcal{P}_\infty(\mathcal{C})$,

(ii) $f_0 = \mathbf{1}$,

(iii) If $p \sim_h q$ in $\mathcal{P}_n(\mathcal{C})$ for some $n \in \mathbb{N}^*$, then $f_p \sim_h f_q$ in $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$.

Thus, from the universal property of $K_0$, one gets a unique group homomorphism

$$\beta_{\mathcal{C}} : K_0(\mathcal{C}) \to K_1\big(S(\mathcal{C})\big)$$

such that $\beta_{\mathcal{C}}([p]_0) = [f_p]_1$ for any $p \in \mathcal{P}_\infty(\mathcal{C})$. The map $\beta_{\mathcal{C}}$ is called *the Bott map.*

If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a unital $*$-homomorphism between unital $C^*$-algebras, then for any $z \in \mathbb{T}$ one has

$$[\widetilde{S(\varphi)}(f_p)](z) = \varphi\big(f_p(z)\big) = f_{\varphi(p)}(z)$$

since $[\widetilde{S(\varphi)}(f)](z) = \varphi\big(f(z)\big)$ for any $f \in M_n\big(\widetilde{S(\mathcal{C})}\big)$. This implies that the diagram

$$\begin{tikzcd} K_0(\mathcal{C}) \arrow[r, "K_0(\varphi)"] \arrow[d, "\beta_{\mathcal{C}}"] & K_0(\mathcal{Q}) \arrow[d, "\beta_{\mathcal{Q}}"] \\ K_1\big(S(\mathcal{C})\big) \arrow[r, "K_1(S(\varphi))"] & K_1\big(S(\mathcal{Q})\big) \end{tikzcd} \tag{7.10}$$

is commutative. This fact is referred to by saying that the Bott map is natural.

Suppose now that $\mathcal{C}$ is a non-unital $C^*$-algebra. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathcal{C}) \arrow[r] \arrow[d, dotted, "\beta_{\mathcal{C}}"] & K_0(\widetilde{\mathcal{C}}) \arrow[r, shift left] \arrow[d, "\beta_{\widetilde{\mathcal{C}}}"] & K_0(\mathbb{C}) \arrow[l, shift left] \arrow[r] \arrow[d, "\beta_{\mathbb{C}}"] & 0 \\ 0 \arrow[r] & K_1\big(S(\mathcal{C})\big) \arrow[r] & K_1\big(S(\widetilde{\mathcal{C}})\big) \arrow[r, shift left] & K_1\big(S(\mathbb{C})\big) \arrow[l, shift left] \arrow[r] & 0\,. \end{tikzcd} \tag{7.11}$$

The right square is commutative because of the commutativity of (7.10). It then follows that there is a unique group homomorphism $\beta_{\mathcal{C}} : K_0(\mathcal{C}) \to K_1\big(S(\mathcal{C})\big)$ making the left square commutative. In addition, a direct computation leads to

$$\beta_{\mathcal{C}}\big([p]_0 - [s(p)]_0\big) = [f_p f_{s(p)}^*]_1 \qquad p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}}). \tag{7.12}$$

It then follows from (7.12) that (7.10) holds also in the non-unital case.

The main result of this section then reads:

**Theorem 7.3.1** (Bott periodicity)**.** *The Bott map $\beta_{\mathcal{C}} : K_0(\mathcal{C}) \to K_1\big(S(\mathcal{C})\big)$ is an isomorphism for any $C^*$-algebra $\mathcal{C}$.*

Note that if $\mathcal{C}$ is non-unital, a diagram chase in (7.11) (or the five lemma) shows that $\beta_{\mathcal{C}}$ is an isomorphism if $\beta_{\widetilde{\mathcal{C}}}$ and $\beta_{\mathbb{C}}$ are isomorphisms. Hence it is sufficient to prove the above theorem for unital $C^*$-algebras. The proof is rather long and technical and will not be reported here. In fact, we shall only state a rather technical lemma from which the main result will be deduced. For more details, we refer to [RLL00, Sec. 11.2] or to [W-O93, Sec. 9.2].

In the following statements, the notation $z^k$ means the map $\mathbb{T} \ni z \mapsto z^k \in \mathbb{T}$ for any natural number $k$.

**Lemma 7.3.2** (Lemma 11.2.13 of [RLL00])**.** *Let $n$ be a natural number.*

*(i) For any $u \in \mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$ there are natural numbers $m \geq n$ and $k$ and an element $p \in \mathcal{P}_m(\mathcal{C})$ such that $(z^k u) \oplus \mathbf{1}_{m-n} \sim_h f_p$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$,*

*(ii) If $p, q$ belong to $\mathcal{P}_n(\mathcal{C})$ with $f_p \sim_h f_q$ in $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$, then there exist a natural number $m \geq n$ and $r \in \mathcal{P}_{m-n}(\mathcal{C})$ such that $p \oplus r \sim_h q \oplus r$ in $\mathcal{P}_m(\mathcal{C})$.*

*Proof of Theorem 7.3.1.* We prove that the Bott map is both surjective and injective.

(i) For a given $g \in K_1\big(S(\mathcal{C})\big)$, let $n \in \mathbb{N}$ and $u \in \mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$ such that $g = [u]_1$. By Lemma 7.3.2.$(i)$, there exist two natural numbers $m \geq n$ and $k$ and an element $p \in \mathcal{P}_m(\mathcal{C})$ such that $(z^k u) \oplus \mathbf{1}_{m-n} \sim_h f_p$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$. By Whitehead's Lemma (Lemma 2.1.4) one also infers that

$$f_{\mathbf{1}_{nk}} = z\mathbf{1}_{nk} \sim_h z^k\mathbf{1}_n \oplus \mathbf{1}_{nk-1} \quad \text{in } \mathcal{U}_{nk}\big(\widetilde{S(\mathcal{C})}\big).$$

As a consequence, one deduces that

$$\begin{aligned}\beta_{\mathcal{C}}\big([p]_0 - [\mathbf{1}_{nk}]_0\big) &= [f_p]_1 - [f_{\mathbf{1}_{nk}}]_1 = [z^k u]_1 - [z^k\mathbf{1}_n]_1 \\ &= [u]_1 + [z^k\mathbf{1}_n]_1 - [z^k\mathbf{1}_n]_1 = [u]_1 = g,\end{aligned}$$

from which one infers the surjectivity of $\beta_{\mathcal{C}}$.

(ii) Let us now consider $g \in K_0(\mathcal{C})$ such that $\beta_{\mathcal{C}}(g) = 0$. Let $n \in \mathbb{N}^*$ and $p, q \in \mathcal{P}_n(\mathcal{C})$ such that $g = [p]_0 - [q]_0$, see Proposition 3.2.4. One then infers that $[f_p]_1 = [f_q]_1$, which implies that $f_p \oplus \mathbf{1}_{m-n} \sim_h f_1 \oplus \mathbf{1}_{m-n}$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$ for some $m \geq n$. Let us then set

$$p_1 := \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_m(\mathcal{C}), \qquad q_1 := \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_m(\mathcal{C}).$$

Then $f_{p_1} = f_p \oplus \mathbf{1}_{m-n}$ and $f_{q_1} = f_q \oplus \mathbf{1}_{m-n}$, and consequently $f_{p_1} \sim_h f_{q_1}$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$. It follows then from Lemma 7.3.2.$(ii)$ that there exist a natural number $k \geq m$ and $r \in \mathcal{P}_{k-m}(\mathcal{C})$ such that $p_1 \oplus r \sim_h q_1 \oplus r$ in $\mathcal{P}_k(\mathcal{C})$. We then conclude that

$$g = [p]_0 - [q]_0 = [p_1 \oplus r]_0 - [q_1 \oplus r]_0 = 0,$$

from which one infers the injectivity of $\beta_{\mathcal{C}}$. □

## 7.4 Applications of Bott periodicity

Bott periodicity makes it possible to compute the $K$-groups of several algebras. First of all, let us state one of its corollary.

**Corollary 7.4.1.** *For any $C^*$-algebras $\mathcal{C}$ and any integer $n$ one has*

$$K_{n+2}(\mathcal{C}) \cong K_n(\mathcal{C}).$$

*Proof.* The case $n = 0$ corresponds precisely to the content of Theorem 7.3.1. The general case follows then by induction on $n$ because

$$K_{n+2}(\mathcal{C}) = K_{n+1}\big(S(\mathcal{C})\big) \cong K_{n-1}\big(S(\mathcal{C})\big) = K_n(\mathcal{C})$$

for any $n \geq 1$. □

**Example 7.4.2.** *We deduce from the previous corollary together with the content of Example 7.2.4 that for any natural number $n$*

$$K_0\big(C_0(\mathbb{R}^n)\big) \cong K_n(\mathbb{C}) \cong \begin{cases} K_0(\mathbb{C}) \cong \mathbb{Z} & n \text{ even} \\ K_1(\mathbb{C}) = \{0\} & n \text{ odd.} \end{cases}$$

*Similarly we have*

$$K_1\big(C_0(\mathbb{R}^n)\big) \cong \begin{cases} \{0\} & n \text{ even} \\ \mathbb{Z} & n \text{ odd.} \end{cases}$$

**Example 7.4.3.** *For any integer $n \geq 0$ consider the $n$-sphere defined by*

$$\mathbb{S}^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

*Clearly, the one-point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$ for any $n \geq 1$, and therefore we have an isomorphism $\widetilde{C_0(\mathbb{R}^n)} \cong C(\mathbb{S}^n)$. In addition, observe from the split exactness of $K_0$, see Proposition 4.3.3, together with the equality $K_0(\mathbb{C}) \cong \mathbb{Z}$, see* (3.12)*, that for any $C^*$-algebra $\mathcal{Q}$ one has*

$$K_0(\widetilde{\mathcal{Q}}) \cong K_0(\mathcal{Q}) \oplus \mathbb{Z}.$$

*It then follows that*

$$K_0\big(C(\mathbb{S}^n)\big) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases} \qquad K_1\big(C(\mathbb{S}^n)\big) \cong \begin{cases} \{0\} & n \text{ even} \\ \mathbb{Z} & n \text{ odd.} \end{cases}$$

*Note that the equality $K_1(\mathcal{C}) \cong K_1(\widetilde{\mathcal{C}})$ of* (5.2) *has been used for the computation of the $K_1$-group.*

# Chapter 8

# The six-term exact sequence

By combining the various results obtained in the previous section we obtain the so-called six-term exact sequence in $K$-theory. In fact, we already know five of the six maps of this sequence. The last map is called *the exponential map* and is constructed from the Bott map composed with the index map $\delta_2$. With this six-term exact sequence, it is possible to compute the $K$-theory of several $C^*$-algebras.

## 8.1 The exponential map and the six-term exact sequence

For any short exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

we define the *exponential map* $\delta_0 : K_0(\mathcal{Q}) \to K_1(\mathcal{J})$ by the composition of the maps

$$K_0(\mathcal{Q}) \xrightarrow{\beta_{\mathcal{Q}}} K_2(\mathcal{Q}) \xrightarrow{\delta_2} K_1(\mathcal{J}),$$

where $\delta_2$ has been defined in the diagram (7.6). In other words if $\bar{\delta}_1$ denotes the index map associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0,$$

then

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\delta_0"] \arrow[d, "\beta_{\mathcal{Q}}"] & K_1(\mathcal{J}) \arrow[d, "\theta_{\mathcal{J}}"] \\ K_1\big(S(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] & K_0\big(S(\mathcal{J})\big) \end{tikzcd} \tag{8.1}$$

is a commutative diagram.

**Theorem 8.1.1** (The six-term exact sequence)**.** *Every short exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*gives rise to the six-term exact sequence*

$$\begin{tikzcd}
K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\
K_0(\mathcal{Q}) \arrow[u, "\delta_0"] & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"] \,.
\end{tikzcd}$$

*Proof.* By Proposition 6.3.3 it only remains to show that this sequence is exact at $K_0(\mathcal{Q})$ and at $K_1(\mathcal{J})$. To see it, consider the diagram

$$\begin{tikzcd}
K_2(\mathcal{C}) \arrow[r, "K_2(\psi)"] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] & K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \\
K_0(\mathcal{C}) \arrow[u, "\beta_{\mathcal{C}}"', "\cong"] \arrow[r, "K_0(\psi)"'] & K_0(\mathcal{Q}) \arrow[u, "\beta_{\mathcal{Q}}"', "\cong"] \arrow[r, "\delta_0"'] & K_1(\mathcal{J}) \arrow[u, "\mathrm{id}"'] \arrow[r, "K_1(\varphi)"'] & K_1(\mathcal{C}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

which is commutative: The left-hand square commutes by naturality of the Bott map, see diagram (7.10), and the center square commutes by the definition of the exponential map. The top row is exact by Proposition 7.2.3, from which one infers that the bottom row is exact as well. □

## 8.2 An explicit description of the exponential map

The exponential map is composed of two natural maps, and therefore is natural as explained in the following statement:

**Proposition 8.2.1.** *The exponential map $\delta_0$ is natural in the following sense: Given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0
\end{tikzcd}$$

*with two short exact sequences of $C^*$-algebras and with three $*$-homomorphisms, the diagram*

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\delta_0"] \arrow[d, "K_0(\beta)"'] & K_1(\mathcal{J}) \arrow[d, "K_1(\gamma)"] \\ K_0(\mathcal{Q}') \arrow[r, "\delta_0'"] & K_1(\mathcal{J}') \ . \end{tikzcd} \tag{8.2}$$

*is commutative, where $\delta_0$ and $\delta_0'$ are the associated exponential maps.*

*Proof.* The diagram (8.2) can be decomposed into two commuting squares

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\beta_{\mathcal{Q}}"] \arrow[d, "K_0(\beta)"] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] \arrow[d, "K_2(\beta)"] & K_1(\mathcal{J}) \arrow[d, "K_1(\gamma)"] \\ K_0(\mathcal{Q}') \arrow[r, "\beta_{\mathcal{Q}'}"'] & K_2(\mathcal{Q}') \arrow[r, "\delta_2'"'] & K_1(\mathcal{J}') \ , \end{tikzcd}$$

as seen in diagrams (7.10) and (7.8). □

The following proposition is somewhat technical but it provides an explicit description of the exponential map and justifies its name.

**Proposition 8.2.2.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

*be a short exact sequence of $C^*$-algebras, and let $\delta_0 : K_0(\mathcal{Q}) \to K_1(\mathcal{J})$ be its associated exponential map. Let $g$ be an element of $K_0(\mathcal{Q})$. Then $\delta_0(g)$ can be computed as follows.*

*(i) Let $p \in \mathcal{P}_n(\widetilde{\mathcal{Q}})$ such that $g = [p]_0 - [s(p)]_0$, and let $a$ be a self-adjoint element in $M_n(\widetilde{\mathcal{C}})$ for which $\tilde{\psi}(a) = p$. Then $\tilde{\varphi}(u) = \exp(2\pi i a)$ for precisely one element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$, and $\delta_0(g) = -[u]_1$.*

*(ii) Suppose that $\mathcal{C}$ is unital, in which case also $\mathcal{Q}$ is unital and $\psi$ is unit preserving. Let $\bar{\varphi} : \widetilde{\mathcal{J}} \to \mathcal{C}$ be given by $\bar{\varphi}(x + \alpha\mathbf{1}) = \varphi(x) + \alpha\mathbf{1}_{\mathcal{C}}$ for any $x \in \mathcal{J}$ and $\alpha \in \mathbb{C}$. Suppose that $g = [p]_0$ for some $p \in \mathcal{P}_n(\mathcal{Q})$, and let $a$ be a self-adjoint element in $M_n(\mathcal{C})$ such that $\psi(a) = p$. Then $\bar{\varphi}(u) = \exp(2\pi i a)$ for precisely one element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$, and $\delta_0(g) = -[u]_1$.*

By the standard picture of $K_0$ provided in Proposition 4.2.1 we can find a projection $p$ as in $(i)$ for any element $g \in K_0(\mathcal{Q})$. In the unital case, any element of $K_0(\mathcal{Q})$ can be described as the difference of two elements of $\mathcal{P}_n(\mathcal{Q})$, as shown in Proposition 3.2.4. In both cases the existence of a self-adjoint lift is provided by Proposition 2.3.1.

*Proof.* (ii) We assume in this proof that $\mathcal{Q} \neq \{0\}$, and hence the map $\bar{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\mathcal{C})$ is injective for any $n \in \mathbb{N}^*$. The image of $\bar{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\mathcal{C})$ consists of those elements $x \in M_n(\mathcal{C})$ such that $\psi(x) \in M_n(\mathbb{C}\mathbf{1}) \subset M_n(\mathcal{Q})$. Then, since

$$\psi\big(\exp(2\pi i a)\big) = \exp\big(2\pi i \psi(a)\big) = \exp(2\pi i p) = \mathbf{1}_n$$

there exists a unique element $u \in M_n(\widetilde{\mathcal{J}})$ such that $\bar{\varphi}(u) = \exp(2\pi i a)$, and since $\bar{\varphi}(u)$ is unitary, one concludes that $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$. By (8.1) we must show that

$$(\bar{\delta}_1 \circ \beta_{\mathcal{Q}})([p]_0) = \theta_{\mathcal{J}}([u^*]_1), \tag{8.3}$$

where $\bar{\delta}_1 : K_1\big(S(\mathcal{Q})\big) \to K_0\big(S(\mathcal{J})\big)$ denotes the index map associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0. \tag{8.4}$$

Note that we shall here use the picture $S(\mathcal{Q}) = C_0\big((0,1); \mathcal{Q}\big)$, for which $M_k\big(\widetilde{S(\mathcal{Q})}\big)$ is identified with the set of all continuous functions $f : [0,1] \to M_k(\mathcal{Q})$ where $f(0) = f(1) \in M_k(\mathbb{C}\mathbf{1}) \subset M_n(\mathcal{Q})$.

In the setting just mentioned, the projection loop $f_p \in \mathcal{U}_n\big(\widetilde{S(\mathcal{Q})}\big)$ associated with the projection $p \in \mathcal{P}_n(\mathcal{Q})$ is given for any $t \in [0,1]$ by

$$f_p(t) = e^{2\pi i t} p + (\mathbf{1}_n - p) = e^{2\pi i t p}.$$

By Lemma 6.1.1 there exists $v \in \mathcal{U}_{2n}\big(\widetilde{S(\mathcal{C})}\big)$ such that $\widetilde{S(\psi)}(v) = \mathrm{diag}(f_p, f_p^*)$. By using a similar identification, one infers that $v : [0,1] \to \mathcal{U}_{2n}(\mathcal{C})$ is a continuous map with $v(0) = v(1)$ belonging to $\mathcal{U}_{2n}(\mathbb{C}\mathbf{1}) \subset \mathcal{U}_{2n}(\mathcal{C})$, and

$$\psi\big(v(t)\big) = \begin{pmatrix} f_p(t) & 0 \\ 0 & f_p(t)^* \end{pmatrix}, \quad \forall t \in [0,1].$$

As $f_p(0) = f_p(1) = \mathbf{1}_n$ one infers that $v(0) = v(1) = \mathbf{1}_{2n}$.

Now, since $a$ is a self-adjoint lift for $p$ in $M_n(\mathcal{C})$, let us set $z(t) = \exp(2\pi i t a)$ for any $t \in [0,1]$. Then $z(t)$ belongs to $\mathcal{U}_n(\mathcal{C})$, the map $t \mapsto z(t)$ is continuous, and $\psi\big(z(t)\big) = f_p(t)$. Hence one gets

$$\psi\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = \mathbf{1}_{2n} \quad \text{and} \quad s\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix}\right) = \mathbf{1}_{2n}.$$

It follows that we can find $w(t)$ in $\mathcal{U}_{2n}(\widetilde{\mathcal{J}})$ with

$$\bar{\varphi}\big(w(t)\big) = v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t) \end{pmatrix} \quad \text{and} \quad s\big(w(t)\big) = \mathbf{1}_{2n}.$$

Now, $t \mapsto w(t)$ is continuous because $\bar{\varphi}$ is isometric, $w(0) = \mathbf{1}_{2n}$, and

$$\bar{\varphi}\big(w(1)\big) = \begin{pmatrix} z(1)^* & 0 \\ 0 & z(1) \end{pmatrix} = \bar{\varphi}\begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix},$$

which shows that $w(1) = \mathrm{diag}(u^*, u)$. By considering the short exact sequence

$$0 \longrightarrow S(\widetilde{\mathcal{J}}) \xrightarrow{\iota} C(\widetilde{\mathcal{J}}) \xrightarrow{\pi(1)} \widetilde{\mathcal{J}} \longrightarrow 0 \tag{8.5}$$

where $\pi(1)$ means the evaluation at the value 1, one infers from Theorem 7.1.3 applied to (8.5) with the unitary element $w \in \mathcal{U}_{2n}\big(\widetilde{C(\widetilde{\mathcal{J}})}\big)$ and the projection $w\left(\begin{smallmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{smallmatrix}\right)w^* \in \mathcal{P}_{2n}\big(\widetilde{S(\widetilde{\mathcal{J}})}\big)$ that

$$\theta_{\mathcal{J}}([u^*]_1) = \left[w\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w^*\right]_0 - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right]_0. \tag{8.6}$$

This corresponds to the r.h.s. of (8.3).

For the l.h.s. of (8.3), recall first that $\beta_{\mathcal{Q}}([p]_0) = [f_p]_1 \in K_1\big(S(\mathcal{Q})\big)$. Note that we also have

$$\bar{\varphi}\left(w(t)\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w(t)^*\right) = v(t)\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}v(t)^*,$$

which implies that

$$\widetilde{S(\varphi)}\left(w\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w^*\right) = v\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}v^*. \tag{8.7}$$

The unitary element $v$ was chosen such that $\widetilde{S(\psi)}(v) = \mathrm{diag}(f_p, f_p^*)$, and so we get from equations (8.7) and from the definition of the index map for the short exact sequence (8.4) that $\bar{\delta}_1([f_p]_1)$ is also equal to the r.h.s. of (8.6). This fact proves (8.3), as expected.

(i) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\iota_{\mathcal{C}}"] & \mathcal{Q} \arrow[r] \arrow[d, "\iota_{\mathcal{Q}}"] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi'"] & \widetilde{\mathcal{C}} \arrow[r, "\psi'"] & \widetilde{\mathcal{Q}} \arrow[r] & 0
\end{tikzcd}$$

where $\varphi' = \iota_{\mathcal{C}} \circ \varphi$ and $\psi' = \tilde{\psi}$. Let $\delta_0'$ be the exponential map associated with the short exact sequence in its lower row. By naturality of the exponential map one gets

$$\begin{aligned}\delta_0\big([p]_0 - [s(p)]_0\big) &= \big(\delta_0' \circ K_0(\iota_{\mathcal{Q}})\big)([p]_0 - [s(p)]_0)\\ &= \delta_0'\big([p]_0 - [s(p)]_0\big) = \delta_0'\big([p]_0\big) - \delta_0'\big([s(p)]_0\big).\end{aligned}$$

The maps $\tilde{\varphi} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{C}}$ and $\overline{\varphi'} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{C}}$ coincide. It follows from $(ii)$ that there is an element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(u) = \overline{\varphi'}(u) = \exp(2\pi i a)$ and that $\delta_0'([p]_0) = -[u]_1$. Since $s(p) = \psi'\big(s(p)\big)$, with $s(p)$ viewed as a scalar projection belonging to $M_n(\widetilde{\mathcal{Q}})$ as well as to $M_n(\widetilde{\mathcal{C}})$, it follows that $[s(p)]_0$ belongs to the image of $K_0(\psi')$ and hence to the kernel of $\delta_0'$. This completes the proof. $\square$

# Chapter 9

# Cyclic cohomology

In Section 3.4 we have shown that any bounded trace $\tau$ on a $C^*$-algebra $\mathcal{C}$ naturally defines a group morphism $K_0(\tau) : K_0(\mathcal{C}) \to \mathbb{C}$ satisfying for any $p \in \mathcal{P}_\infty(\mathcal{C})$

$$K_0(\tau)([p]_0) = \tau(p).$$

However, many important $C^*$-algebras do not possess such a bounded trace, and one still would like to extract some numerical invariants from their $K$-groups. One solution to this problem can be obtained by using cyclic cohomology, which is the subject of this chapter. Our main references are the books [Con94, Chap. III], [Kha13, Chap. 3], [MN08, Chap. 5] and the paper [KS04].

## 9.1 Basic definitions

Let $\mathcal{A}$ be a complex associative algebra, and for any $n \in \mathbb{N}$ let $C^n(\mathcal{A})$ denote the set of $(n+1)$-linear functionals on $\mathcal{A}$. The elements $\eta \in C^n(\mathcal{A})$ are called *$n$-cochains*.

**Definition 9.1.1.** *An element $\eta \in C^n(\mathcal{A})$ is* cyclic *if it satisfies for each $a_0, \dots, a_n \in \mathcal{A}$:*

$$\eta(a_1, \dots, a_n, a_0) = (-1)^n \eta(a_0, \dots, a_n) \ .$$

*The set of all cyclic $n$-cochains is denoted by $C^n_\lambda(\mathcal{A})$.*

For any $\eta \in C^n(\mathcal{A})$ let us also define $\mathtt{b}\eta$ by

$$[\mathtt{b}\eta](a_0, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \eta(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \eta(a_{n+1} a_0, \dots, a_n). \tag{9.1}$$

Then one has:

**Lemma 9.1.2.** *The space of cyclic cochains is invariant under the action of* $\mathtt{b}$, i.e. *for any $n \in \mathbb{N}$ one has*

$$\mathtt{b}\, C^n_\lambda(\mathcal{A}) \subset C^{n+1}_\lambda(\mathcal{A}).$$

*Proof.* Define the operator $\lambda : C^n(\mathcal{A}) \to C^n(\mathcal{A})$ and $\mathtt{b}' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ by

$$\big(\lambda\eta\big)(a_0, \dots, a_n) := (-1)^n \eta(a_n, a_0, \dots, a_{n-1}), \tag{9.2}$$

$$\big(\mathtt{b}'\eta\big)(a_0, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \eta(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}). \tag{9.3}$$

One readily checks that $(\mathbf{1} - \lambda)\mathtt{b} = \mathtt{b}'(\mathbf{1} - \lambda)$ and that

$$C^n_\lambda(\mathcal{A}) = \mathsf{Ker}(\mathbf{1} - \lambda),$$

which imply the statement of the lemma. □

As a consequence of the previous lemma one can consider the complex

$$C^0_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} C^1_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} C^2_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} \dots$$

which is called *the cyclic complex of* $\mathcal{A}$. Note that the property $\mathtt{b}^2 = 0$, necessary for the next definition, is quite standard in this framework and can be checked quite easily.

**Definition 9.1.3.** *(i) An element $\eta \in C^n_\lambda(\mathcal{A})$ satisfying $\mathtt{b}\eta = 0$ is called a* cyclic $n$-cocycle*, and the set of all cyclic $n$-cocycles is denoted by $Z^n_\lambda(\mathcal{A})$,*

*(ii) An element $\eta \in C^n_\lambda(\mathcal{A})$ with $\eta \in \mathtt{b}\big(C^{n-1}_\lambda(\mathcal{A})\big)$ is called* a cyclic $n$-coboundary*, and the set all cyclic $n$-coboundaries is denoted by $B^n_\lambda(\mathcal{A})$,*

*(iii) The cohomology of the cyclic complex of $\mathcal{A}$ is called* the cyclic cohomology of $\mathcal{A}$*, and more precisely $HC^n(\mathcal{A}) := Z^n_\lambda(\mathcal{A})/B^n_\lambda(\mathcal{A})$ for any $n \in \mathbb{N}$. The elements of $HC^n(\mathcal{A})$ are called* the classes of cohomology*.*

Note that a cyclic $n$-cocycle $\eta$ is simply a $(n+1)$-linear functional on $\mathcal{A}$ which satisfies the two conditions

$$\mathtt{b}\eta = 0 \quad \text{and} \quad (\mathbf{1} - \lambda)\eta = 0, \tag{9.4}$$

where $\mathtt{b}$ has been introduced in (9.1) and $\lambda$ in (9.2). Observe also that the equality

$$Z^0_\lambda(\mathcal{A}) = HC^0(\mathcal{A}) \tag{9.5}$$

holds, and that this space corresponds to the set of all traces on $\mathcal{A}$.

**Example 9.1.4.** *Let us consider the case $\mathcal{A} = \mathbb{C}$. Then, any $\eta \in C^n_\lambda(\mathbb{C})$ is completely determined by its value $\eta(1, \dots, 1)$. In addition, by the cyclicity property one infers that $\eta = 0$ whenever $n$ is odd. Hence the cyclic complex of $\mathbb{C}$ is simply given by*

$$\mathbb{C} \to 0 \to \mathbb{C} \to \dots$$

*from which one deduces that for any $k \in \mathbb{N}$:*

$$HC^{2k}(\mathbb{C}) \cong \mathbb{C} \quad \textit{and} \quad HC^{2k+1}(\mathbb{C}) = 0.$$

Let us now consider another convenient way of looking at cyclic $n$-cocycles in terms of characters of a graded differential algebra over $\mathcal{A}$. For that purpose, we first recall that a graded differential algebra $(\Omega, \mathtt{d})$ is a graded algebra

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \dots \ ,$$

with each $\Omega^j$ an associative algebra over $\mathbb{C}$, together with a map $\mathtt{d} : \Omega^j \to \Omega^{j+1}$ which satisfy

(i) $w_j w_k \in \Omega^{j+k}$ for any elements $w_j \in \Omega^j$ and $w_k \in \Omega^k$,

(ii) For any homogeneous elements $w_1, w_2$ and if $\deg(w)$ denotes the degree of a homogeneous element $w$:

$$\mathtt{d}(w_1 w_2) = (\mathtt{d}w_1)w_2 + (-1)^{\deg(w_1)} w_1(\mathtt{d}w_2), \tag{9.6}$$

(iii) $\mathtt{d}^2 = 0$.

**Definition 9.1.5.** *A* closed graded trace of dimension $n$ *on the graded differential algebra* $(\Omega, \mathtt{d})$ *is a linear functional* $\int : \Omega^n \to \mathbb{C}$ *satisfying for any* $w \in \Omega^{n-1}$

$$\int \mathtt{d}w = 0, \tag{9.7}$$

*and for any* $w_j \in \Omega^j$, $w_k \in \Omega^k$ *such that* $j + k = n$*:*

$$\int w_j w_k = (-1)^{jk} \int w_k w_j \ . \tag{9.8}$$

With these definitions at hand, we can now set:

**Definition 9.1.6.** *A* $n$-cycle *is a triple* $(\Omega, \mathtt{d}, \int)$ *consisting in a graded differential algebra* $(\Omega, \mathtt{d})$ *together with a closed graded trace* $\int$ *of dimension* $n$*. This* $n$*-cycle is* over the algebra $\mathcal{A}$ *if in addition there exists an algebra homomorphism* $\rho : \mathcal{A} \to \Omega^0$*.*

Given a $n$-cycle over the algebra $\mathcal{A}$, one defines the corresponding *character* $\eta$ by

$$\eta(a_0, \dots, a_n) := \int \rho(a_0)\, \mathtt{d}\rho(a_1) \dots \mathtt{d}\rho(a_n) \in \mathbb{C} \tag{9.9}$$

for any $a_0, \dots, a_n \in \mathcal{A}$. The link between such characters and cyclic $n$-cocycles can now be proved.

**Proposition 9.1.7.** *Any* $n$*-cycle* $(\Omega, \mathtt{d}, \int)$ *over the algebra* $\mathcal{A}$ *defines a cyclic* $n$*-cocycle through its character* $\eta$ *given by* (9.9)*.*

For simplicity we shall drop the homomorphism $\rho : \mathcal{A} \to \Omega^0$ in the sequel, or equivalently identify $\mathcal{A}$ with $\rho(\mathcal{A})$ in $\Omega^0$.

*Proof.* With the convention mentioned before, one can rewrite (9.9) as

$$\eta(a_0, \dots, a_n) := \int a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_n.$$

By using Leibnitz rule as recalled in (9.6) and the graded trace property as mentioned in (9.8) one then gets

$$\begin{aligned}
&\big(\mathtt{b}\eta\big)(a_0, \dots, a_{n+1}) \\
&= \int a_0 \, a_1 \mathtt{d}a_2 \dots \mathtt{d}a_{n+1} + \sum_{j=1}^{n} (-1)^j \int a_0 \, \mathtt{d}a_1 \dots \mathtt{d}(a_j a_{j+1}) \dots \mathtt{d}a_{n+1} \\
&+ (-1)^{n+1} \int a_{n+1} a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_n \\
&= (-1)^n \int a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_n \, a_{n+1} + (-1)^{n+1} \int a_{n+1} a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_n \\
&= 0.
\end{aligned}$$

On the other hand, by using the closeness property recalled in (9.7) one also gets

$$\begin{aligned}
\big((\mathbf{1} - \lambda)\eta\big)(a_0, \dots, a_n) &= \int a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_n - (-1)^n \int a_n \, \mathtt{d}a_0 \dots \mathtt{d}a_{n-1} \\
&= (-1)^{n-1} \int \mathtt{d}\big(a_n a_0 \, \mathtt{d}a_1 \dots \mathtt{d}a_{n-1}\big) \\
&= 0.
\end{aligned}$$

Since the two conditions of (9.4) are satisfied, the statement follows. □

Let us end this section with some examples of cyclic cocycles.

**Example 9.1.8.** *Let $\mathcal{A}$ be a complex and associative algebra, and let $\tau$ be a linear functional on $\mathcal{A}$ with the tracial property,* i.e. *$\tau(ab) = \tau(ba)$ for any $a, b \in \mathcal{A}$. Assume also that $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation on $\mathcal{A}$ such that $\tau\big(\delta(a)\big) = 0$ for any $a \in \mathcal{A}$. Then the map*

$$\mathcal{A} \times \mathcal{A} \ni (a_0, a_1) \mapsto \tau\big(a_0 \, \delta(a_1)\big) \in \mathbb{C}$$

*is a cyclic 1-cocycle on $\mathcal{A}$.*

*More generally, if $\delta_1, \dots, \delta_n$ are mutually commuting derivations on $\mathcal{A}$ which satisfy $\tau\big(\delta_j(a)\big) = 0$ for any $a \in \mathcal{A}$ and $j \in \{1, 2, \dots, n\}$, then for any $a_0, \dots, a_n \in \mathcal{A}$*

$$\eta(a_0, a_1, \dots, a_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \tau\big(a_0 \, \delta_{\sigma(1)}(a_1) \dots \delta_{\sigma(n)}(a_n)\big)$$

*defines a cyclic $n$-cocycle on $\mathcal{A}$. Here we have used the standard notation $S_n$ for the permutation group of $n$ elements.*

**Example 9.1.9.** *Let $\mathcal{M}$ be an oriented compact and smooth manifold of dimension $n$, and set $\mathcal{A} := C^\infty(\mathcal{M})$. Then the map $\eta$ defined for $f_0, f_1, \ldots, f_n \in \mathcal{A}$ by*

$$\eta(f_0, f_1, \ldots, f_n) := \int_{\mathcal{M}} f_0 \, \mathtt{d}f_1 \wedge \cdots \wedge \mathtt{d}f_n$$

*is a cyclic $n$-cocycle on $\mathcal{A}$.*

**Example 9.1.10.** *In $\mathcal{H} := L^2(\mathbb{R}^n)$ consider the algebra $\mathcal{K}^\infty$ of smooth integral operators defined for any $u \in \mathcal{H}$ by*

$$[Au](x) = \int_{\mathbb{R}^n} a(x,y)u(y)\,\mathrm{d}y$$

*for some $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ (the Schwartz space on $\mathbb{R}^n \times \mathbb{R}^n$). Define the derivation $\delta_1, \ldots, \delta_{2n}$ on $\mathcal{K}^\infty$ by*

$$\delta_{2j}(A) := [X_j, A], \quad \delta_{2j-1}(A) := [D_j, A], \qquad j \in \{1, 2, \ldots, n\}$$

*with $X_j$ the operator of multiplication by the variable $x_j$ and $D_j := -i\frac{\partial}{\partial x_j}$. Then one easily checks that $\delta_j\delta_k = \delta_k\delta_j$, and that $\mathrm{Tr}\big(\delta_j(A)\big) = 0$, where $\mathrm{Tr}(A) = \int_{\mathbb{R}^n} a(x,x)\,\mathrm{d}x$. Then the map $\eta$ defined for any $a_0, a_1, \ldots, a_{2n} \in \mathcal{K}^\infty$ by*

$$\eta(a_0, a_1, \ldots, a_{2n}) := \frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{Tr}\big(a_0\, \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(2n)}(a_{2n})\big)$$

*is a cyclic $2n$-cocycle on $\mathcal{K}^\infty$.*

## 9.2 Cup product in cyclic cohomology

Starting from two classes of cohomology, our aim in this section is to construct a new class of cohomology. The key ingredient is the *cup product* which is going to be defined below.

First of all, let $(\Omega, \mathtt{d}, \int)$ be a $n$-cycle over $\mathcal{A}$ and let $(\Omega', \mathtt{d}', \int')$ be a $n'$-cycle over a second complex associative algebra $\mathcal{A}'$. The corresponding algebra homomorphisms are denoted by $\rho : \mathcal{A} \to \Omega^0$ and $\rho' : \mathcal{A}' \to \Omega'^0$. The graded differential tensor product algebra $\big(\Omega \otimes \Omega', \mathtt{d} \otimes \mathtt{d}'\big)$ is then defined by

$$(\Omega \otimes \Omega')^\ell := \oplus_{j+k=\ell}\, \Omega^j \otimes \Omega'^k$$

and

$$\mathtt{d} \otimes \mathtt{d}'(w \otimes w') := (\mathtt{d}w) \otimes w' + (-1)^{\deg(w)} w \otimes (\mathtt{d}'w').$$

We also set for $w \in \Omega^n$ and $w' \in \Omega'^{n'}$

$$\int^{\otimes} \omega \otimes \omega' := \int \omega \int' \omega'$$

which is a closed graded trace of dimension $n+n'$ on $\big(\Omega\otimes\Omega', \mathtt{d}\otimes\mathtt{d}'\big)$. Finally, the map

$$\rho\otimes\rho' : \mathcal{A}\otimes\mathcal{A}' \to (\Omega\otimes\Omega')^0 \equiv \Omega^0\otimes\Omega'^0$$

defines an algebra homomorphism which makes $\big(\Omega\otimes\Omega', \mathtt{d}\otimes\mathtt{d}', \int^{\otimes}\big)$ a $(n+n')$-cycle over $\mathcal{A}\otimes\mathcal{A}'$.

In a vague sense, the above construction associates with the two characters $\eta$ and $\eta'$, defined by (9.9), a new character obtained from the $(n+n')$-cycle over $\mathcal{A}\otimes\mathcal{A}'$. Since characters define cyclic $n$-cocyles, one has obtained a new $(n+n')$-cocycle in terms of a $n$-cycle and a $n'$-cycle. However, a deeper result can be obtained.

For that purpose, let us now denote by $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ the universal graded differential algebra over $\mathcal{A}$. We shall not recall its construction here, but refer to [Con85, p. 98-99], [Con94, p. 185-186] or [GVF01, Sec. 8.1]. For the time being, let us simply mention that any $(n+1)$-linear functional $\eta$ on $\mathcal{A}$, *i.e.* any $\eta\in C^n(\mathcal{A})$, defines a linear functional $\hat{\eta}$ on $\Omega(\mathcal{A})^n$ by the formula

$$\hat{\eta}(a_0\,\mathtt{d}a_1\ldots\mathtt{d}a_n) := \eta(a_0,\ldots,a_n) \qquad \forall a_0,\ldots,a_n\in\mathcal{A}. \tag{9.10}$$

Also, in terms of $\Omega(\mathcal{A})$ a generalization of Proposition 9.1.7 reads:

**Proposition 9.2.1.** *Let $\eta$ be a $(n+1)$-linear functional on $\mathcal{A}$. Then the following conditions are equivalent:*

*(i) There is a $n$-cycle $(\Omega, \mathtt{d}, \int)$ over $\mathcal{A}$, with $\rho:\mathcal{A}\to\Omega^0$ the corresponding algebra homomorphism, such that*

$$\eta(a_0,\ldots,a_n) := \int \rho(a_0)\,\mathtt{d}\rho(a_1)\ldots\mathtt{d}\rho(a_n) \qquad \forall a_0,\ldots,a_n,$$

*(ii) There exists a closed graded trace $\hat{\eta}$ of dimension $n$ on $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ such that*

$$\eta(a_0,\ldots,a_n) = \hat{\eta}(a_0\,\mathtt{d}a_1\ldots\mathtt{d}a_n) \qquad \forall a_0,\ldots,a_n,$$

*(iii) $\eta$ is a cyclic $n$-cocycle.*

Note that the proof of this proposition is not very difficult, once a good description of $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ has been provided, see [Con94, Prop. III.1.4].

Let us now come to the cup product. In general, the graded differential algebra $\big(\Omega(\mathcal{A}\otimes\mathcal{A}'), \mathtt{d}\big)$ and the graded differential tensor product algebra $\big(\Omega(\mathcal{A})\otimes\Omega(\mathcal{A}'), \mathtt{d}\otimes\mathtt{d}'\big)$ are not equal. However, from the universal property of $\big(\Omega(\mathcal{A}\otimes\mathcal{A}'), \mathtt{d}\big)$ one infers that there exists a natural homomorphism

$$\pi : \Omega(\mathcal{A}\otimes\mathcal{A}') \to \Omega(\mathcal{A})\otimes\Omega(\mathcal{A}').$$

This homomorphism plays a role in the definition of the cup product.

For any $\eta \in C^n(\mathcal{A})$ and $\eta' \in C^{n'}(\mathcal{A}')$, recall that $\hat{\eta}$ and $\widehat{\eta'}$ are respectively linear functionals on $\Omega(\mathcal{A})^n$ and on $\Omega(\mathcal{A}')^{n'}$. One then defines the cup product $\eta \# \eta' \in C^{n+n'}(\mathcal{A} \otimes \mathcal{A}')$ by the equality

$$\widehat{(\eta \# \eta')} = \big(\hat{\eta} \otimes \widehat{\eta'}\big) \circ \pi.$$

Some properties of this product are gathered in the following statement.

**Proposition 9.2.2.** *(i) The cup product $\eta \otimes \eta' \mapsto \eta \# \eta'$ defines a homomorphism*

$$HC^n(\mathcal{A}) \otimes HC^{n'}(\mathcal{A}') \to HC^{n+n'}(\mathcal{A} \otimes \mathcal{A}'),$$

*(ii) The character of the tensor product of two cycles is the cup product of their characters.*

The proof is provided in [Con94, Thm. III.1.12]. Let us however mention that its main ingredients are the equivalences recalled Proposition 9.2.1 as well as the subsequent diagram (for the point $(ii)$). For it, let us consider again a $n$-cycle $\big(\Omega, \mathtt{d}, \int\big)$ over $\mathcal{A}$ and a $n'$-cycle $\big(\Omega', \mathtt{d}', \int'\big)$ over $\mathcal{A}'$, with respective algebra homomorphisms $\rho : \mathcal{A} \to \Omega^0$ and $\rho' : \mathcal{A}' \to \Omega'^0$. Since the universal graded differential algebra $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ is generated by $\mathcal{A}$, there exists a unique extension of $\rho$ to a morphism of differential graded algebras

$$\tilde{\rho} : \Omega(\mathcal{A}) \to \Omega.$$

As a consequence, there also exist two additional algebra homomorphisms

$$\widetilde{\rho'} : \Omega(\mathcal{A}') \to \Omega' \quad \text{and} \quad \widetilde{\rho \otimes \rho'} : \Omega(\mathcal{A} \otimes \mathcal{A}') \to \Omega \otimes \Omega'.$$

Then, by the universal property of $\Omega(\mathcal{A} \otimes \mathcal{A}')$ the following diagram is commutative:

$$\begin{tikzcd}
\Omega(\mathcal{A} \otimes \mathcal{A}') \arrow[r, "\pi"] \arrow[dr, "\widetilde{\rho \otimes \rho'}"'] & \Omega(\mathcal{A}) \otimes \Omega(\mathcal{A}') \arrow[d, "\tilde{\rho} \otimes \widetilde{\rho'}"] \\
& \Omega \otimes \Omega' \ .
\end{tikzcd}$$

**Example 9.2.3.** *Let $\tau$ be a trace on an algebra $\mathcal{B}$,* i.e. $\tau \in HC^0(\mathcal{B})$. *Then the map*

$$HC^n(\mathcal{A}) \ni \eta \mapsto \eta \# \tau \ni HC^n(\mathcal{A} \otimes \mathcal{B})$$

*is explicitly given on product elements by*

$$[\eta \# \tau](a_0 \otimes b_0, \ldots, a_n \otimes b_n) = \eta(a_0, \ldots, a_n)\, \tau(b_0 b_1 \ldots b_n).$$

*As a special example, let $\mathcal{B} = M_k(\mathbb{C})$ and let $\tau = \mathrm{tr}$ be the usual traces on matrices. In this case, the cup product defines a map*

$$HC^n(\mathcal{A}) \ni \eta \mapsto \eta \# \mathrm{tr} \in \ HC^n\big(M_k(\mathcal{A})\big).$$

*On more general elements* $A^0, \dots, A^n \in M_k(\mathcal{A})$ *and if one denotes by* $A^\ell_{ij}$ *the component* $(i,j)$ *of* $A^\ell$ *for* $\ell \in \{0, 1, \dots, n\}$*, then one has*

$$[\eta \# \mathrm{tr}](A^0, A^1, \dots, A^n) = \sum_{j_0,\dots,j_n=1}^{k} \eta\big(A^0_{j_0 j_1}, A^1_{j_1 j_2}, \dots, A^n_{j_n j_0}\big).$$

We end this section with the introduction of the periodicity operator $S$. This operator is obtained by considering the special case $\mathcal{A}' = \mathbb{C}$ in the above construction. As mentioned in Example 9.1.4, any cyclic 2-cocycle $\eta'$ on $\mathbb{C}$ is uniquely defined by $\eta'(1,1,1)$ which we chose to be equal to 1[1]. Since $\mathcal{A} \otimes \mathbb{C} = \mathcal{A}$ one gets a map of degree 2 in cyclic cohomology, commonly denoted by S:

$$S : HC^n(\mathcal{A}) \ni \eta \mapsto S\eta := \eta \# \eta' \ \in HC^{n+2}(\mathcal{A}).$$

A computation involving the universal graded differential algebra $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ as presented in [Con94, Corol. III.1.13] leads then to the explicit formula

$$\begin{aligned}[S\eta](a_0, a_1, \dots, a_{n+2}) = {} & \hat{\eta}\big(a_0\, a_1\, a_2\, \mathtt{d}a_3 \dots \mathtt{d}a_{n+2}\big) + \hat{\eta}\big(a_0\, \mathtt{d}a_1\, (a_2\, a_3)\, \mathtt{d}a_4 \dots \mathtt{d}a_{n+2}\big) \\ & + \sum_{i=3}^{n} \hat{\eta}\big(a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_{i-1}\, (a_i\, a_{i+1})\, \mathtt{d}a_{i+2} \dots \mathtt{d}a_{n+2}\big) \\ & + \hat{\eta}\big(a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_n\, (a_{n+1}\, a_{n+2})\big),\end{aligned}$$

where $\hat{\eta}$ is the linear functional on $\Omega(\mathcal{A})^n$ defined by $\eta$.

## 9.3 Unbounded derivations

In the previous two sections, only algebraic manipulations were considered, and it was not necessary for $\mathcal{A}$ to have a topology. We shall now consider topological vector spaces and topological algebras, the $C^*$-condition will appear only at the end of the game. However, interesting cocycles are often only defined on dense subalgebras, and this naturally leads us to the definition of unbounded operators/derivations and unbounded traces. In this section we focus on the former ones.

Given two topological vector spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ over $\mathbb{C}$ [2], a *linear operator from* $\mathcal{B}_1$ *to* $\mathcal{B}_2$ consists in a pair $\big(T, \mathsf{Dom}(T)\big)$, where $\mathsf{Dom}(T) \subset \mathcal{B}_1$ is a linear subspace and $T : \mathsf{Dom}(T) \to \mathcal{B}_2$ is a linear map. Note that most of the time one simply speaks about the operator $T$, but a domain $\mathsf{Dom}(T)$ is always attached to it. This operator is *densely defined* if $\mathsf{Dom}(T)$ is dense in $\mathcal{B}_1$. One also says that this operator is *closable* if the closure of its graph

$$\mathsf{Graph}(T) := \big\{(\xi, T\xi) \in \mathcal{B}_1 \times \mathcal{B}_2 \mid \xi \in \mathsf{Dom}(T)\big\}$$

[1] The conventions about the normalization differ from one reference to another one. For example, in [Con94] it is assumed that $\eta'(1,1,1) = 1$ but in [Con85] the convention $\eta'(1,1,1) = 2\pi i$ is taken.

[2] A topological vector space $X$ over $\mathbb{C}$ is a vector space which is endowed with a topology such that the addition $X \times X \to X$ and the scalar multiplication $\mathbb{C} \times X \to X$ are continuous.

does not contain an element of the form $(0, \xi')$ with $\xi' \neq 0$. In this case, the closure $\overline{\mathsf{Graph}(T)}$ is equal to $\mathsf{Graph}(\bar{T})$ for a unique linear operator $\bar{T}$ called *the closure of* $T$.

For the definition of the dual operator, let us assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are locally convex topological vector spaces[3], and let $\mathcal{B}_j^*$ denote the strong dual of $\mathcal{B}_j$[4], see [Yos65, Sec. IV.7] for the details. Then the *dual operator* $\big(T^*, \mathsf{Dom}(T^*)\big)$ of a densely defined unbounded operator $\big(T, \mathsf{Dom}(T)\big)$ is defined as the unbounded operator from $\mathcal{B}_2^*$ to $\mathcal{B}_1^*$ such that

$$\begin{aligned} \mathsf{Dom}(T^*) &:= \big\{\zeta^* \in \mathcal{B}_2^* \mid \exists \xi^* \in \mathcal{B}_1^* \text{ s.t. } \langle T\xi, \zeta^*\rangle = \langle \xi, \xi^*\rangle \quad \forall \xi \in \mathsf{Dom}(T)\big\} \\ T^*\zeta^* &:= \xi^*. \end{aligned}$$

Note that we use the same notation $\langle \cdot, \cdot \rangle$ for various duality relations. Observe also that the equality $\langle \xi, T^*\zeta^*\rangle := \langle T\xi, \zeta^*\rangle$ holds for all $\xi \in \mathsf{Dom}(T)$ and $\zeta^* \in \mathsf{Dom}(T^*)$. We finally mention that the dual operator of a densely defined operator $T$ is always closed. In the context of Banach spaces, the dual operator is often called *the adjoint operator*.

Let us now define the general notion of a derivation, and then consider more precisely unbounded derivations.

**Definition 9.3.1.** *Let $\mathcal{B}_1$ be a Banach algebra and let $\mathcal{B}_2$ be a topological vector space which is also a $\mathcal{B}_1$-bimodule*[5]*. Then, an unbounded operator $\delta$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ is called* a derivation *if $\mathsf{Dom}(\delta)$ is a subalgebra of $\mathcal{B}_1$ and if for any $a, b \in \mathsf{Dom}(\delta)$:*

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The following result is then standard, see [Con94, Lem. III.6.2].

**Lemma 9.3.2.** *Let $\mathcal{B}_1$ be a unital Banach algebra and let $\mathcal{B}_2$ be a Banach $\mathcal{B}_1$-bimodule*[6] *satisfying for any $b, b' \in \mathcal{B}_1$ and $\xi \in \mathcal{B}_2$*

$$\|b\xi b'\| \leq \|b\| \, \|\xi\| \, \|b'\|. \tag{9.11}$$

*Let $\delta$ be a densely defined and closable derivation from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then its closure $\bar{\delta}$ is still a derivation, and its domain $\mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}_1$ stable under holomorphic*

[3] A locally convex topological vector space is a vector space together with a family of semi-norms $\{p_{\mathfrak{a}}\}$ which defines its topology. Any Banach space is a locally convex topological vector space.

[4] The strong dual of a locally convex topological vector space $X$ consists in the set of all continuous linear functionals on $X$ endowed with the bounded convergence topology. This topology is defined by the family of semi-norms of the form $p^*(T) \equiv p_B^*(T) := \sup_{x \in B} |T(x)|$ with $T$ a continuous linear functional on $X$ and $B$ any bounded set of $X$.

[5] A $\mathcal{B}$-bimodule is an Abelian group $X$ together with a multiplication on the right and on the left by elements of a ring $\mathcal{B}$ (satisfying some natural conditions) and for which the equality $(a\xi)b = a(\xi b)$ holds for any $a, b \in \mathcal{B}$ and any $\xi \in X$.

[6] A Banach $\mathcal{B}$-bimodule is a Banach space $X$ which is also a $\mathcal{B}$-bimodule with $\mathcal{B}$ a Banach algebra $\mathcal{B}$, and for which the multiplication on the left and on the right also satisfy $\|a\xi\| \leq c\|a\|\|\xi\|$ and $\|\xi b\| \leq c\|b\|\|\xi\|$ for some $c > 0$, all $\xi \in X$ and any $a, b \in \mathcal{B}$.

*functional calculus*[7].

Let us illustrate the use of the previous definition with two examples. For that purpose, observe that for any Banach algebra $\mathcal{B}$ its dual space $\mathcal{B}^*$ can be viewed as a $\mathcal{B}$-bimodule by the relation

$$\langle b\xi b', a\rangle := \langle \xi, b'ab\rangle \quad \forall a, b, b' \in \mathcal{B} \text{ and } \xi \in \mathcal{B}^*.$$

Moreover the relation (9.11) holds, namely $\|b\xi b'\| \leq \|b\|\,\|\xi\|\,\|b'\|$.

**Lemma 9.3.3.** *Let $\mathcal{B}$ be a unital $C^*$-algebra, and let $\delta$ be a densely defined derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule $\mathcal{B}^*$. Assume in addition that $\mathbf{1}_{\mathcal{B}}$ belongs to $\mathsf{Dom}(\delta^*)$ (in $\mathcal{B}^{**}$). Then,*

*(i) $\tau := \delta^*(\mathbf{1}_{\mathcal{B}}) \in \mathcal{B}^*$ defines a trace on $\mathcal{B}$,*

*(ii) The map $K_0(\tau) : K_0(\mathcal{B}) \to \mathbb{C}$ corresponds to the $0$-map.*

Let us mention that this statement holds for more general Banach algebras and the $C^*$-property does not play any role. However, since the $K$-groups have been introduced only for $C^*$-algebras, we restrict the statement to this framework. Unfortunately, the proof below uses the $K$-groups for more general algebras, and therefore is only partially understandable in our setting.

*Proof.* i) For any $a, b \in \mathsf{Dom}(\delta)$ one first infers that:

$$\begin{aligned}\tau(ab) &:= \langle \delta^*(\mathbf{1}_{\mathcal{B}}), ab\rangle = \langle \mathbf{1}_{\mathcal{B}}, \delta(ab)\rangle = \langle \mathbf{1}_{\mathcal{B}}, \delta(a)b\rangle + \langle \mathbf{1}_{\mathcal{B}}, a\delta(b)\rangle \\ &= \langle b, \delta(a)\rangle + \langle a, \delta(b)\rangle = \langle a, \delta(b)\rangle + \langle b, \delta(a)\rangle \\ &= \langle \mathbf{1}_{\mathcal{B}}, \delta(b)a\rangle + \langle \mathbf{1}_{\mathcal{B}}, b\delta(a)\rangle = \langle \mathbf{1}_{\mathcal{B}}, \delta(ba)\rangle = \langle \delta^*(\mathbf{1}_{\mathcal{B}}), ba\rangle = \tau(ba).\end{aligned}$$

Since $\tau$ is a bounded functional on $\mathcal{B}$, one deduces from the density of $\mathsf{Dom}(\delta)$ in $\mathcal{B}$ that the equality $\tau(ab) = \tau(ba)$ holds for any $a, b \in \mathcal{B}$.

ii) Let $a \in \mathsf{Dom}(\delta)$. The equality

$$\langle a, \delta(b)\rangle = \tau(ab) - \langle b, \delta(a)\rangle$$

[7] In order to define the notion of *stable under holomorphic functional calculus*, let us consider a unital Banach algebra $\mathcal{B}$, an element $a \in \mathcal{B}$ and let $f$ be a holomorphic function defined on a neighbourhood $O$ of $\sigma(a)$. We then define

$$f(a) := \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}\,\mathrm{d}z \ \in \mathcal{B}$$

with $\gamma$ a closed curve of finite length and without self-intersection in $O$ encircling $\sigma(a)$ only once and counterclockwise. By holomorphy of $f$ this integral is independent of the choice of $\gamma$, and for any fixed $a$, the map $f \mapsto f(a)$ defines a functional calculus, called *the holomorphic functional calculus of $a$*. In this setting, if $\mathcal{A}$ is a unital and dense subalgebra of a unital Banach algebra $\mathcal{B}$ one says that $\mathcal{A}$ is stable under holomorphic functional calculus if $f(a) \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and $f$ is a holomorphic function in a neighbourhood of $\sigma(a)$.

valid for any $b \in \mathsf{Dom}(\delta)$ shows that $\mathsf{Dom}(\delta) \subset \mathsf{Dom}(\delta^*)$, with $\delta^*(a) = \tau(a\,\cdot) - \delta(a)$ for any $a \in \mathsf{Dom}(\delta)$. Since $\delta^*$ is closed operator and $\tau(a\,\cdot)$ is bounded operator, one infers that $\delta$ is a closable operator from $\mathcal{B}$ to $\mathcal{B}^*$, whose extension is denoted by $\bar{\delta}$. By Lemma 9.3.2 one deduces that its domain $\mathcal{A} := \mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}$ which is stable under holomorphic functional calculus. A consequence of this stability is that there exists an isomorphism between $K_0(\mathcal{A})$ (which has not been defined in these lecture notes) and $K_0(\mathcal{B})$, see [Con94, III.App.C] and the references mentioned there.

The next step consists in showing that the homomorphism $K_0(\tau)$ corresponds to the 0-map on all elements of $p \in M_n(\mathcal{A})$ satisfying $p^2 = p$. For simplicity, let us choose $n = 1$ (in the general case, use $\tau \otimes \mathrm{tr}$ on $\mathcal{A} \otimes M_n(\mathbb{C})$ instead of $\tau$ on $\mathcal{A}$). Then, for any $p \in \mathcal{A}$ satisfying $p^2 = p$ one has

$$\tau(p) = \tau(p^2) = \langle \delta^*(\mathbf{1}_{\mathcal{B}}), p^2 \rangle = 2\langle p, \delta(p) \rangle.$$

But one also has

$$\langle p, \delta(p) \rangle = \langle p, \delta(p^2) \rangle = 2\langle p, \delta(p) \rangle$$

from which one infers that $\tau(p) = 0$. Since $p \in \mathcal{A}$ is an arbitrary idempotent, the statement then follows once a suitable description of $K_0(\mathcal{A})$ has been provided. □

The next statement has a similar flavor. It deals with the notion of a 1-trace, and motivates the introduction of more general $n$-traces in the next section. Again, let us mention that the following statement holds for more general Banach algebras.

**Proposition 9.3.4.** *Let $\mathcal{B}$ be a unital $C^*$-algebra, and let $\delta$ be a densely defined derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule $\mathcal{B}^*$ satisfying for all $a, b \in \mathsf{Dom}(\delta)$*

$$\langle \delta(a), b \rangle = -\langle \delta(b), a \rangle. \tag{9.12}$$

*Then,*

*(i) $\delta$ is closable, with its closure denoted by $\bar{\delta}$,*

*(ii) There exists a unique map $\varphi : K_1(\mathcal{B}) \to \mathbb{C}$ such that for any $v \in \mathcal{GL}_n\big(\mathsf{Dom}(\bar{\delta})\big)$ one has*

$$\varphi([w(v)]_1) = \langle \bar{\delta}(v), v^{-1} \rangle$$

*where $w(v) \in \mathcal{U}_n(\mathcal{B})$ is defined by $w(v) = v|v|^{-1}$, as explained in Proposition 2.1.8.*

*Proof.* The result about the closability of $\delta$ is standard. It follows from the closeness of $\delta^*$ and from the fact that $\delta^* = -\delta$ on $\mathsf{Dom}(\delta)$, which means that $\delta$ is a skew-symmetric operator and $-\delta^*$ is a closed extension of it. As a consequence, $\delta$ is closable with closure denoted by $\bar{\delta}$.

Since $\delta$ is closable, it follows from Lemma 9.3.2 that $\bar{\delta}$ is also a derivation, and its domain $\mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}$ stable under holomorphic functional calculus. Let us denote by $\mathcal{A}$ this subalgebra. Let us mention two important properties of this stability, and refer to [CMR07, Prop. 2.58 & Thm. 2.60] for a proof of these properties

in a more general context: Firstly, if $v_1, v_2 \in \mathcal{GL}_n(\mathcal{A})$ are homotopic in $\mathcal{GL}_n(\mathcal{B})$ then they are also homotopic in $\mathcal{GL}_n(\mathcal{A})$. Secondly, there exists an isomorphism between $K_1(\mathcal{A})$ (which has not been defined in these lecture notes) and $K_1(\mathcal{B})$.

Let us now consider a piecewise affine path $t \mapsto v(t)$ in $\mathcal{GL}_n(\mathcal{A})$ and observe that the function $f(t) := \langle \bar{\delta}\big(v(t)\big), v(t)^{-1}\rangle$ is constant. Indeed, its derivative with respect to $t$ satisfies

$$\begin{aligned} f'(t) &= \langle \bar{\delta}\big(v'(t)\big), v(t)^{-1}\rangle - \langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\rangle \\ &= -\langle \bar{\delta}\big(v(t)^{-1}\big), v'(t)\rangle - \langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\rangle \\ &= \langle v(t)^{-1}\bar{\delta}\big(v(t)\big)v(t)^{-1}, v'(t)\rangle - \langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\rangle \\ &= 0. \end{aligned}$$

In other words, the expression $\langle \bar{\delta}(v), v^{-1}\rangle$ is constant on piecewise affine paths in $\mathcal{GL}_n(\mathcal{A})$, and the statement follows once a suitable description of $K_1(\mathcal{A})$ has been provided. □

## 9.4 Higher traces

We shall now insert the results obtained in the previous section in a more general framework. Before stating the main definition of this section, let us extend slightly the validity of (9.10). First of all, recall that if $\mathcal{A}$ is any associative and unital[8] algebra over $\mathbb{C}$ we set $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ for the corresponding universal graded differential algebra over $\mathcal{A}$. In this setting, the following equality holds for any $a_j, a_k \in \mathcal{A} \subset \Omega(\mathcal{A})^0$

$$(\mathtt{d}a_j)a_k = \mathtt{d}(a_j\, a_k) - a_j\, \mathtt{d}(a_k),$$

from which one infers the equality valid for any $a_0, \dots, a_k, a \in \mathcal{A}$:

$$\begin{aligned} &a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_k\, a \\ &= a_0\, \mathtt{d}a_1 \dots \mathtt{d}(a_k\, a) - a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_{k-1}\, a_k\, \mathtt{d}a \\ &= (-1)^k a_0\, a_1\, \mathtt{d}a_2 \dots \mathtt{d}a_k\, \mathtt{d}a + \sum_{j=1}^{k-1} (-1)^{k-j} a_0\, \mathtt{d}a_1 \dots \mathtt{d}(a_j\, a_{j+1}) \dots \mathtt{d}a_k\, \mathtt{d}a \\ &\quad + a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_{k-1}\, \mathtt{d}(a_k\, a). \end{aligned}$$

For any $(n+1)$-linear functional $\eta$ on $\mathcal{A}$, the previous equalities allow us to give a meaning to expressions of the form

$$\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)$$

[8]For simplicity we assume $\mathcal{A}$ to be unital but the following construction also holds in the non-unital case.

for any $a_0, \dots, a_n, x_1 \dots, x_n \in \mathcal{A}$. Indeed, if one first replaces $x_j\, \mathtt{d}a_j$ with $\mathtt{d}(x_j\, a_j) - (\mathtt{d}x_j)a_j$ and then moves $a_j$ to the left with the above equalities, one gets an equality of the form

$$x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n) = \sum_\ell (-1)^{m_\ell} b_{0\ell}\, \mathtt{d}b_{1\ell}\, \mathtt{d}b_{2\ell} \dots \mathtt{d}b_{n\ell}$$

for some $b_{j\ell} \in \mathcal{A}$ and $m_\ell \in \mathbb{N}$. By the linearity of $\eta$ one can finally set

$$\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big) = \sum_\ell (-1)^{m_\ell} \eta(b_{0\ell}, b_{1\ell}, \dots, b_{n\ell}).$$

For example, for $n = 2$ one has

$$\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2)\big) = \eta(x_1, a_1\, x_2, a_2) - \eta(x_1\, a_1, x_2, a_2).$$

With these notations, the main definition of this section then reads:

**Definition 9.4.1.** *Let $\mathcal{B}$ be a unital Banach algebra and $n \in \mathbb{N}$. A $n$-*trace *on $\mathcal{B}$ is a cyclic $n$-cocycle $\eta$ on a dense subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that for any $a_1, \dots, a_n \in \mathcal{A}$ there exists $c = c(a_1, \dots, a_n) > 0$ with*

$$\big|\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)\big| \leq c\, \|x_1\|\, \|x_2\| \dots \|x_n\| \qquad \forall x_1, \dots, x_n \in \mathcal{A}. \tag{9.13}$$

Our aim will be to show that when $\mathcal{B}$ is a unital $C^*$-algebra, any such $n$-traces on $\mathcal{B}$ determines a map from $K_i(\mathcal{B})$ to $\mathbb{C}$, for $i \in \{0, 1\}$. For that purpose, lots of preliminary works are necessary.

**Remark 9.4.2.** *For fixed $a_1, \dots, a_n$, condition* (9.13) *allows one to extend the multi-linear functional defined on $\mathcal{A}^n$ to a bounded multi-linear functional on $\mathcal{B}^n$. The values taken by this functional on elements of $\mathcal{B}^n$ are obtained by a limiting process. For this extension, we shall freely write $\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)$ with $x_j \in \mathcal{B}$.*

**Remark 9.4.3.** *Let us observe that the two examples of the previous section fit into the definition of a* 0 *and of a* 1*-trace. Indeed, in the framework of Lemma 9.3.3 one already inferred that $\tau := \delta^*(\mathbf{1}_{\mathcal{B}})$ is a trace on $\mathcal{B}$, and thus a cyclic* 0*-cocycle. In addition, the estimate $|\tau(x_1)| \leq \|\tau\|\, \|x_1\|$ holds for any $x_1 \in \mathcal{B}$. In the framework of Proposition 9.3.4 one can set $\eta(a_0, a_1) := \langle \delta(a_0), a_1 \rangle$. By taking the property* (9.12) *into account, as well as the definition of the $\mathcal{B}$-bimodule $\mathcal{B}^*$, one gets for any $a_0, a_1, a_2 \in \mathsf{Dom}(\delta)$*

$$\begin{aligned}
[\mathtt{b}\eta](a_0, a_1, a_2) &= \eta(a_0 a_1, a_2) - \eta(a_0, a_1 a_2) + \eta(a_2 a_0, a_1) \\
&= \langle \delta(a_0 a_1), a_2 \rangle - \langle \delta(a_0), a_1 a_2 \rangle + \langle \delta(a_2 a_0), a_1 \rangle \\
&= -\langle \delta(a_2), a_0 a_1 \rangle - \langle \delta(a_0), a_1 a_2 \rangle + \langle \delta(a_2) a_0, a_1 \rangle + \langle a_2 \delta(a_0), a_1 \rangle \\
&\quad - \langle \delta(a_2), a_0 a_1 \rangle - \langle \delta(a_0), a_1 a_2 \rangle + \langle \delta(a_2), a_0 a_1 \rangle + \langle \delta(a_0), a_1 a_2 \rangle \\
&= 0
\end{aligned}$$

*from which one easily infers that $\eta$ is a cyclic* 1*-cocycle on $\mathcal{A} := \mathsf{Dom}(\delta)$. In addition, one has for any $a_1, x_1 \in \mathcal{A}$*

$$|\hat{\eta}(x_1 \mathtt{d}a_1)| = |\eta(x_1, a_1)| = |\langle \delta(x_1), a_1 \rangle| = |\langle \delta(a_1), x_1 \rangle| \leq \|\delta(a_1)\|\, \|x_1\|$$

*which corresponds to* (9.13) *in this special case.*

Let us now denote by $E$ the vector space of all $(2n-1)$-linear functionals

$$\varphi : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n \text{ times}} \times \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{(n-1) \text{ times}} \to \mathbb{C}$$

which are continuous in the $\mathcal{B}$-variables. We endow $E$ with the family of semi-norms $(p_{\mathfrak{a}})_{\mathfrak{a} \in \mathcal{A}^{n-1}}$ defined by

$$p_{\mathfrak{a}}(\varphi) \equiv p_{(a_1,\ldots,a_{n-1})}(\varphi) := \sup_{x_j \in \mathcal{B}, \|x_j\| \leq 1} |\varphi(x_1, \ldots, x_n, a_1, \ldots, a_{n-1})|.$$

With this family of semi-norms, $E$ becomes a locally convex topological vector space. In addition, for any $\varphi \in E$ and any $x \in \mathcal{B}$ we also set

$$\begin{aligned}
[x\varphi](x_1, \ldots, x_n, a_1, \ldots, a_{n-1}) &:= \varphi(x_1\, x, x_2, \ldots, x_n, a_1, \ldots, a_{n-1}), \\
[\varphi x](x_1, \ldots, x_n, a_1, \ldots, a_{n-1}) &:= \varphi(x_1, x\, x_2, \ldots, x_n, a_1, \ldots, a_{n-1}).
\end{aligned}$$

Endowed with this additional structure, one has obtained a $\mathcal{B}$-bimodule $E$. Note finally that for any $\mathfrak{a} \in \mathcal{A}^{n-1}$ one has

$$p_{\mathfrak{a}}(x\varphi) \leq p_{\mathfrak{a}}(\varphi)\|x\| \quad \text{and} \quad p_{\mathfrak{a}}(\varphi x) \leq p_{\mathfrak{a}}(\varphi)\|x\|. \tag{9.14}$$

Based on the previous definition, the following lemma can now be proved. Recall that the notion of a derivation has been introduced in Definition 9.3.1.

**Lemma 9.4.4.** *Let $\mathcal{B}$ be a unital Banach algebra, $n \in \mathbb{N}$ and let $\eta$ be a $n$-trace on $\mathcal{B}$ defined on a dense subalgebra denoted by $\mathcal{A}$. For any $a \in \mathcal{A}$ let*

$$\delta(a) : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n \text{ times}} \times \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{(n-1) \text{ times}} \to \mathbb{C}$$

*be defined by*

$$[\delta(a)](x_1, \ldots, x_n, a_2, \ldots, a_n) := \hat{\eta}\big((x_1\, \mathtt{d}a)(x_2\, \mathtt{d}a_2) \ldots (x_n\, \mathtt{d}a_n)\big).$$

*Then:*

*(i) $\delta$ is a derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule $E$, with $\mathsf{Dom}(\delta) = \mathcal{A}$,*

*(ii) $\delta$ is closable, when $\mathcal{B}$ is endowed with its norm topology and $E$ with the topology of simple convergence.*

*Proof.* i) It clearly follows from the definition of the $n$-trace that $\delta(a)$ is an element of $E$. In addition, for any $a, b \in \mathcal{A}$ and since $\mathtt{d}(ab) = (\mathtt{d}a)b + a(\mathtt{d}b)$ one has

$$\begin{aligned}
&[\delta(ab)](x_1, \ldots, x_n, a_2, \ldots, a_n) \\
&= \hat{\eta}\big((x_1\, \mathtt{d}(ab))(x_2\, \mathtt{d}a_2) \ldots (x_n\, \mathtt{d}a_n)\big) \\
&= \hat{\eta}\big((x_1\, \mathtt{d}a)(b x_2\, \mathtt{d}a_2) \ldots (x_n\, \mathtt{d}a_n)\big) + \hat{\eta}\big((x_1\, a\, \mathtt{d}b)(x_2\, \mathtt{d}a_2) \ldots (x_n\, \mathtt{d}a_n)\big) \\
&= [\delta(a)](x_1, b x_2, x_3, \ldots, x_n, a_2, \ldots, a_n) + [\delta(b)](x_1\, a, x_2, \ldots, x_n, a_2, \ldots, a_n) \\
&= [\delta(a)b](x_1, \ldots, x_n, a_2, \ldots, a_n) + [a\delta(b)](x_1, \ldots, x_n, a_2, \ldots, a_n).
\end{aligned}$$

By setting $\mathsf{Dom}(\delta) := \mathcal{A}$, one infers that $\delta$ is a derivation from $\mathcal{B}$ to $E$.

ii) By the definition of the topologies on $\mathcal{B}$ and $E$, we have to show that for any $\{a_\nu\} \subset \mathcal{A}$ with $\|a_\nu\| \to 0$ and $\delta(a_\nu) \to \varphi \in E$ in the weak sense[9] as $\nu \to \infty$, then $\varphi = 0$. By density of $\mathcal{A}$ in $\mathcal{B}$, it is sufficient to show that $\varphi(x_1, \dots, x_n, a_2, \dots, a_n) = 0$ for any $x_j, a_j \in \mathcal{A}$.

By assumption on $\varphi$ and by taking into account the properties of the closed graded trace $\hat{\eta}$ of dimension $n$ on $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ one gets

$$\begin{aligned}
&\big|\varphi(x_1, \dots, x_n, a_2, \dots, a_n)\big| \\
&= \lim_{\nu\to\infty} \big|\hat{\eta}\big((x_1\,\mathtt{d}a_\nu)(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)\big| \\
&= \lim_{\nu\to\infty} \big|\hat{\eta}\big(\mathtt{d}a_\nu(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\big| \\
&= \lim_{\nu\to\infty} \Big|\hat{\eta}\Big(\mathtt{d}\big(a_\nu(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big) - \hat{\eta}\Big(a_\nu\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&= \lim_{\nu\to\infty} \Big|\hat{\eta}\Big(a_\nu\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&\le c \lim_{\nu\to\infty} \|a_\nu\| \\
&= 0.
\end{aligned}$$

Note that we had to choose $x_j \in \mathcal{A}$ instead of $x_j \in \mathcal{B}$ in order to give a meaning to the expression $\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)$. □

Our aim is now to extend the domain of the derivation $\delta$, in a way similar to the one obtained in the less general framework of Lemma 9.3.2. For that purpose, let us call *$\delta$-bounded* a subset $X$ of $\mathcal{A}$ whenever there exist finitely many $c_1, \dots, c_k \in \mathcal{A}$ such that for all $a \in X$:

$$p_{\mathfrak{a}}\big(\delta(a)\big) \le \sup_{j\in\{1,\dots,k\}} p_{\mathfrak{a}}\big(\delta(c_j)\big) \qquad \forall \mathfrak{a} \in \mathcal{A}^{n-1}.$$

We can then consider the subset $\mathscr{B}$ of $\mathcal{B}$ defined as follows: $a \in \mathscr{B}$ if there exists a $\delta$-bounded sequence $\{a_m\}_m \subset \mathcal{A}$ converging to $a$ in $\mathcal{B}$. In other words, $a \in \mathscr{B}$ if there exist $\{a_m\}_m \subset \mathcal{A}$ and finitely many $c_1, \dots, c_k \in \mathcal{A}$ such that $a_m \to a$ in $\mathcal{B}$ and

$$\sup_m p_{\mathfrak{a}}\big(\delta(a_m)\big) \le \sup_{j\in\{1,\dots,k\}} p_{\mathfrak{a}}\big(\delta(c_k)\big) \qquad \forall \mathfrak{a} \in \mathcal{A}^{n-1}.$$

By this assumption the sequence $\delta(a_m)$ is then bounded in $E$, and for any $x_j, a_j \in \mathcal{A}$ one has

$$\begin{aligned}
&\Big|[\delta(a_m)](x_1, \dots, x_n, a_2, \dots, a_n) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&= \Big|\hat{\eta}\big((x_1\,\mathtt{d}a_m)(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&= \Big| - \hat{\eta}\Big(a_m\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&\le c\,\|a_m - a\|
\end{aligned}$$

[9] Let us emphasize that $\nu$ belongs to a directed set, a sequence might not be general enough.

which converges to 0 as $m \to \infty$. As a consequence, one gets that $\delta(a_m)$ weakly converges to $\delta(a)$, independently of the choice of the sequence $\{a_m\}_m$, with

$$[\delta(a)](x_1, \dots, x_n, a_2, \dots, a_n) = -\hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big) \qquad \forall x_j, a_j \in \mathcal{A}.$$

Note that by construction one has $p_{\mathfrak{a}}\big(\delta(a)\big) \leq \sup_k p_{\mathfrak{a}}\big(\delta(c_k)\big)$ for all $\mathfrak{a} \in \mathcal{A}^{n-1}$.

The following result then holds:

**Lemma 9.4.5.** *(i) $\mathscr{B}$ is a dense subalgebra of $\mathcal{B}$ containing $\mathcal{A}$,*

*(ii) For any $q \in \mathbb{N}^*$, $M_q(\mathscr{B})$ is stable under holomorphic functional calculus.*

We only provide the proof of the first statement. For the second one, we refer to [Con86, Lem. 2.3].

*Proof.* Let $a, a' \in \mathscr{B}$ with two sequences $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{A}$ satisfying $a_n \to a$ and $a'_n \to a'$. Clearly, $a_n a'_n \to aa'$ in $\mathcal{B}$, and in addition one has for any $\mathfrak{a} \in \mathcal{A}^{n-1}$

$$p_{\mathfrak{a}}\big(\delta(a_n a'_n)\big) = p_{\mathfrak{a}}\big(\delta(a_n)a'_n + a_n\delta(a'_n)\big) \leq c\Big(p_{\mathfrak{a}}\big(\delta(a_n)\big) + p_{\mathfrak{a}}\big(\delta(a'_n)\big)\Big) \leq \sup_{j\in\{1,\dots,k\}} p_{\mathfrak{a}}\big(\delta(c_j)\big)$$

for some $c < \infty$ independent of $n$ and a finite family of elements $c_1, \dots, c_k \in \mathcal{A}$. Note that (9.14) has been used for the first inequality. By definition of $\mathscr{B}$, the previous computation means that $aa' \in \mathscr{B}$, which is thus an algebra.

For the density, it is sufficient to observe that $\mathcal{A} \subset \mathscr{B}$, and to recall that $\mathcal{A}$ is dense in $\mathcal{B}$. □

In the next statement we mention an argument which has already been used in the proofs of Lemma 9.3.3 and of Proposition 9.3.4. Unfortunately, its content is not fully understandable in the context of these lecture notes since $K_0(\mathscr{B})$ and $K_1(\mathscr{B})$ have not been defined ($\mathscr{B}$ is not a $C^*$-algebra but a dense subalgebra which is closed under holomorphic functional calculus).

**Proposition 9.4.6.** *Let $\mathcal{B}$ be a unital $C^*$-algebra endowed with a $n$-trace, and let $\mathscr{B}$ be the dense subalgebra defined above.*

*(i) The inclusion $\mathscr{B} \subset \mathcal{B}$ defines an isomorphism of $K_0(\mathscr{B})$ with $K_0(\mathcal{B})$,*

*(ii) The inclusion $\mathscr{B} \subset \mathcal{B}$ defines an isomorphism of $K_1(\mathscr{B})$ with $K_1(\mathcal{B})$.*

**Remark 9.4.7.** *With this remark, we would like to emphasize that in the definition of $\mathscr{B}$ the $n$-trace plays a significant role. In that respect, the existence of a cyclic cocycle precedes the construction of a suitably dense subalgebra of $\mathcal{B}$. In other words in order to extract information from the $K$-groups, one should not first define any specific dense subalgebra but look for a $n$-trace, and then define a suitable subalgebra stable under holomorphic functional calculus. Note that a similar observation is also made on page 113 of [MN08].*

We end up this section with two technical lemmas. All this material will be useful in the next section and for the main theorem of this chapter. For that purpose and for any finite family $a_1, \dots, a_n \in \mathcal{A}$ we set

$$C(a_1, \dots, a_n) := p_{\mathfrak{a}}\big(\delta(a_1)\big) \quad \text{with} \quad \mathfrak{a} = (a_2, \dots, a_n) \in \mathcal{A}^{n-1}.$$

**Lemma 9.4.8.** *(i) For any $a_1, \dots, a_n \in \mathcal{A}$ the value $C(a_1, \dots, a_n)$ is invariant under cyclic permutations of its arguments,*

*(ii) If $X$ is a $\delta$-bounded subset of $\mathcal{A}$ then*

$$\sup_{a_1,\dots,a_n \in X} C(a_1, \dots, a_n) < \infty.$$

*Proof.* i) By definition one has

$$C(a_1, \dots, a_n) = \sup_{x_j \in \mathcal{B}, \|x_j\| \leq 1} \big|\hat{\eta}\big((x_1\, \mathtt{d}a_1)(x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)\big|.$$

Since $\hat{\eta}$ is a closed graded trace of dimension $n$ on $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ it follows from the permutation property (9.8) that $C(a_1, \dots, a_n)$ is invariant under cyclic permutations of its arguments.

ii) Let $c_1, \dots, c_k \in \mathcal{A}$ such that $p_{\mathfrak{a}}\big(\delta(a)\big) \leq \sup_j p_{\mathfrak{a}}\big(\delta(c_j)\big)$ for all $\mathfrak{a} \in \mathcal{A}^{n-1}$ and $a \in X$. Now, observe that

$$C(a_1, \dots, a_n) = p_{\mathfrak{a}}\big(\delta(a_1)\big) \leq \sup_j p_{\mathfrak{a}}\big(\delta(c_j)\big) = \sup_j C(c_j, a_2, \dots, a_n).$$

By the same argument and by taking point (i) into account one gets that for $a_2 \in X$

$$\begin{aligned} C(c_j, a_2, \dots, a_n) &= C(a_2, \dots, a_n, c_1) \\ &= p_{(a_3,\dots,a_n,c_1)}\big(\delta(a_2)\big) \\ &\leq \sup_{j'} p_{(a_3,\dots,a_n,c_1)}\big(\delta(c_{j'})\big) \\ &\leq \sup_{j'} C(c_{j'}, a_3, \dots, a_n, c_j). \end{aligned}$$

By iteration one finally infers that if $a_i \in X$ for any $i \in \{1, \dots, n\}$ then

$$C(a_1, \dots, a_n) \leq \sup_{j_1,\dots,j_n} C(c_{j_1}, \dots, c_{j_n}) < \infty$$

as claimed. □

**Lemma 9.4.9.** *Let $x_1, \dots, x_n \in \mathcal{B}$ and consider $n$ $\delta$-bounded convergent sequences $\{a_{j,m}\}_m \subset \mathcal{A}$ with $\lim_m a_{j,m} = a_{j,\infty}$ in $\mathcal{B}$, for any $j \in \{1, \dots, n\}$. Then the sequence*

$$m \mapsto \hat{\eta}\big((x_1\, \mathtt{d}a_{1,m})(x_2\, \mathtt{d}a_{2,m}) \dots (x_n\, \mathtt{d}a_{n,m})\big)$$

*converges to a limit which depends only on $a_{1,\infty}, \dots, a_{n,\infty}, x_1, \dots, x_n$.*

Observe that by definition of $\mathscr{B}$, the above assumptions imply that $a_{1,\infty}, \dots, a_{n,\infty}$ belong to $\mathscr{B}$.

*Proof.* Let us denote by $X$ the set of all $a_{j,m}$ which is $\delta$-bounded by assumption. By Lemma 9.4.8 the family of all multi-linear functionals $\varphi$ on $\mathcal{B}^n$ defined by

$$\varphi(x_1, \dots, x_n) := \hat{\eta}\big((x_1\, \mathtt{d}a_1)(x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)$$

with $a_j \in X$ is a bounded set. Let us then define

$$\varphi_m(x_1, \dots, x_n) := \hat{\eta}\big((x_1\, \mathtt{d}a_{1,m})(x_2\, \mathtt{d}a_{2,m}) \dots (x_n\, \mathtt{d}a_{n,m})\big).$$

In order to show the simple convergence of the sequence $\varphi_m$ we can assume that $x_1, \dots, x_n$ belong to the dense subset $\mathcal{A}$ of $\mathcal{B}$. We also slightly enlarge $X$ by defining $X' = X \cup \{x_1, \dots, x_n\}$. By applying again Lemma 9.4.8.$(ii)$ with $X'$ instead of $X$, and by taking the equality

$$\hat{\eta}\big(x_1\, \mathtt{d}a\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big) = -\hat{\eta}\Big(a\mathtt{d}\big((x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)x_1\big)\Big)$$

into account, one gets that there exists a constant $c > 0$ (which depends on $x_1, \dots, x_n$ but not on $a, a_2, \dots, a_n$) such that

$$\Big|\hat{\eta}\Big(a\, \mathtt{d}\big((x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)x_1\big)\Big)\Big| \leq c\, \|a\| \quad \forall a_j \in X.$$

Similarly, we also get that

$$\big|\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big) - \hat{\eta}\big(x_1\, \mathtt{d}a_1'(x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)\big| \leq c\, \|a_1 - a_1'\|$$

for any $a_1, \dots, a_n, a_1' \in X$. By using cyclic permutations one also infers that

$$\big|\hat{\eta}\big(x_1\, \mathtt{d}a_1\, (x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big) - \hat{\eta}\big(x_1\, \mathtt{d}a_1'(x_2\, \mathtt{d}a_2') \dots (x_n\, \mathtt{d}a_n')\big)\big| \leq c \sum_{j=1}^{n} \|a_j - a_j'\|$$

for any $a_1, \dots, a_n, a_1', \dots, a_j' \in X$.

We deduce from the previous estimates that the sequence $\varphi_m(x_1, \dots, x_n)$ is a Cauchy sequence, and that the limit of this sequence does not depend upon the choice of the $\delta$-bounded sequences $\{a_{j,m}\}$ converging to $a_{j,\infty}$. □

## 9.5 Pairing of cyclic cohomology with $K$-theory

In this section we finally pair cyclic cohomology with $K$-theory. We shall first deal with even cyclic cocycle.

For any $n \in \mathbb{N}$ let us consider a $2n$-cyclic cocycle $\eta$ on an associative complex unital algebra $\mathcal{A}$, *i.e.* $\eta \in Z^{2n}_{\lambda}(\mathcal{A})$. As seen in section 9.2 on the cup product, one can define

an element $\eta\#\mathrm{tr} \in Z^{2n}_\lambda\big(M_q(\mathcal{A})\big)$ for any $q \in \mathbb{N}^*$. Let also $p$ be an idempotent in $M_q(\mathcal{A})$, *i.e.* an element $p \in M_q(\mathcal{A})$ satisfying $p^2 = p$. With these two ingredients one can define and study the expression

$$\frac{1}{n!}\,[\eta\#\mathrm{tr}](\underbrace{p, p, \ldots, p}_{2n+1 \text{ terms}}). \tag{9.15}$$

Let us first observe that this expression depends only on the cyclic cohomology class of $\eta$ in $HC^{2n}(\mathcal{A})$. By considering directly the algebra $M_q(\mathcal{A})$ instead of $\mathcal{A}$, it is sufficient to concentrate on the special case $q = 1$. Thus, let us assume that $\eta = \mathtt{b}\psi$ for some $\psi \in C^{2n-1}_\lambda(\mathcal{A})$, or in another words let us assume that $\eta$ is a coboundary. Then by the definition of $\mathtt{b}$ one has

$$\begin{aligned}
\eta(p, \ldots, p) &= \mathtt{b}\psi(p, \ldots, p) \\
&= \sum_{j=0}^{2n-1} (-1)^j \psi(p, \ldots, p, \ldots, p) + \psi(p, \ldots, p) \\
&= \psi(p, \ldots, p) \\
&= 0,
\end{aligned}$$

where the cyclicity of $\psi$ has been used for the last equality.

Our next aim is to state and prove a result related to the periodicity operator $S$ introduced at the end of Section 9.2. Let us stress once again that for the next statement we choose $\eta'(1, 1, 1) = 1$ for the normalization of $\eta' \in Z^2_\lambda(\mathbb{C})$, other conventions are also used in the literature. As in the computation made before, we can restrict our attention of the case $q = 1$.

**Lemma 9.5.1.** *For any $\eta \in Z^{2n}_\lambda(\mathcal{A})$ and for any idempotent $p \in \mathcal{A}$ one has*

$$[S\eta](\underbrace{p, p, \ldots, p}_{2n+3 \text{ terms}}) = (n+1)\,\eta(\underbrace{p, p, \ldots, p}_{2n+1 \text{ terms}}).$$

*Proof.* By taking the expression mentioned at the end of Section 9.2 into account one gets

$$\begin{aligned}
[S\eta](p, \ldots, p) = \hat{\eta}\big(p\,\mathtt{d}p \ldots \mathtt{d}p\big) + \hat{\eta}\big(p\,\mathtt{d}p\,p\,\mathtt{d}p \ldots \mathtt{d}p\big) \\
+ \sum_{i=3}^{2n} \hat{\eta}\big(p\,\mathtt{d}p \ldots \mathtt{d}p\,p\,\mathtt{d}p \ldots \mathtt{d}p\big) + \hat{\eta}\big(p\,\mathtt{d}p \ldots \mathtt{d}p\,p\big).
\end{aligned}$$

Then, since $\mathtt{d}p\,p = \mathtt{d}p - p\,\mathtt{d}p$, one easily infers that nearly half of the terms in the previous sum simply cancel out and one obtains that

$$[S\eta](p, \ldots, p) = (n+1)\hat{\eta}\big(p\,\mathtt{d}p \ldots \mathtt{d}p\big) = (n+1)\eta(p, \ldots, p),$$

which corresponds to the statement. □

Our last aim with respect to the expression (9.15) is to show that it is invariant under the conjugation of $p$ by invertible elements. As before, we can restrict our attention to the case $q = 1$.

**Lemma 9.5.2.** *For any $\eta \in Z^{2n}_\lambda(\mathcal{A})$, any idempotent $p \in \mathcal{A}$ and any $v \in \mathcal{GL}(\mathcal{A})$ one has*

$$\eta(p, \dots, p) = \eta(vpv^{-1}, \dots, vpv^{-1}).$$

The proof of this statement is based on the following observation and result. If $\mathcal{A}, \mathcal{B}$ are associative complex algebras and if $\rho : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism, then it induces a morphism $\rho^* : C^n_\lambda(\mathcal{B}) \to C^n_\lambda(\mathcal{A})$ defined for any $a_0, \dots, a_n \in \mathcal{A}$ by

$$[\rho^*\eta](a_0, \dots, a_n) = \eta\big(\rho(a_0), \dots, \rho(a_n)\big).$$

As a consequence, it also induces a map $\rho^* : HC^n(\mathcal{B}) \to HC^n(\mathcal{A})$. This map depends only on the equivalence class of $\rho$ modulo inner automorphisms, as mentioned in the next statement (see [Con94, Prop. III.1.8] for its proof):

**Proposition 9.5.3.** *Let $v \in \mathcal{GL}(\mathcal{A})$, and set $\theta(a) := vav^{-1}$ for any $a \in \mathcal{A}$ for the corresponding inner automorphism. Then the induced map $\theta^* : HC^n(\mathcal{A}) \to HC^n(\mathcal{A})$ is the identity for any $n \in \mathbb{N}$.*

*Proof of Lemma 9.5.2.* With the notations of the previous statement, it is enough to observe that

$$\eta(vpv^{-1}, \dots, vpv^{-1}) = \eta\big(\theta(p), \dots, \theta(p)\big) = [\theta^*\eta](p, \dots, p) = \eta(p, \dots, p).$$

Note that for the last equality, the result of the previous proposition has been used together with the fact that (9.15) depends only on the cohomology class of $\eta$. □

**Remark 9.5.4.** *An alternative approach for the proof of Lemma 9.5.2 is provided in [Kha13, p. 169-170]. However, even if this alternative approach might look more intuitive (the derivative of a certain expression computed along a smooth path of idempotents is zero), it works only in some suitable topological algebras. In the previous proof, no topological structure is involved.*

All the information obtained so far on the expression (9.15) can now be gathered in a single statement. As before, the only missing bit of information is about the definition of $K_0(\mathcal{A})$ in this framework.

**Proposition 9.5.5.** *For any $q \in \mathbb{N}^*$ and $n \in \mathbb{N}$, the following map is bilinear*

$$K_0(\mathcal{A}) \times HC^{2n}(\mathcal{A}) \ni \ ([p]_0, [\eta]) \mapsto \langle [p]_0, [\eta] \rangle := \frac{1}{n!}\,[\eta \# \mathrm{tr}](p, \dots, p) \ \in \mathbb{C}, \tag{9.16}$$

*with $p \in M_q(\mathcal{A})$ an idempotent and with $\eta \in Z^{2n}_\lambda(\mathcal{A})$. In addition, the following properties hold:*

*(i)* $\langle [p]_0, [S\eta]\rangle = \langle [p]_0, [\eta]\rangle$,

*(ii)* *If* $\eta, \eta'$ *are even cyclic cocycles on the algebras* $\mathcal{A}, \mathcal{A}'$, *then for any idempotents* $p \in \mathcal{A}$ *and* $p' \in \mathcal{A}'$ *one has*

$$\big\langle [p \otimes p']_0, [\eta \# \eta']\big\rangle = \langle [p]_0, [\eta]\rangle \, \langle [p']_0, [\eta']\rangle$$

*and a similar formula holds in a matricial version.*

For odd cyclic cocycles, a similar construction and similar statements can be proved. We proceed in a similar way. For $n \in \mathbb{N}$ let $\eta$ be a $(2n+1)$-cyclic cocycle over $\mathcal{A}$, *i.e.* $\eta \in Z^{2n+1}_\lambda(\mathcal{A})$, with $\mathcal{A}$ an associative complex unital algebra. For any $q \in \mathbb{N}^*$ and $v \in \mathcal{GL}_q(\mathcal{A})$ we shall study the expression

$$[\eta \# \mathrm{tr}](\underbrace{v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \ldots, v^{-1} - \mathbf{1}, v - \mathbf{1}}_{2n+2 \text{ terms}}). \tag{9.17}$$

Before studying in details this expression, let us recall that the $K_1$-theory of a $C^*$-algebra $\mathcal{C}$ was computed from the equivalence classes of $\mathcal{U}_\infty(\widetilde{\mathcal{C}})$, see Definition 5.1.2. For the $K_1$-theory of $\mathcal{A}$, the addition of a unit is also part of the construction. Note also that at the price of changing the algebra $\mathcal{A}$ to $M_q(\mathcal{A})$, we can restrict our attention to the case $q = 1$ in the following developments.

As a preliminary step, let $\widetilde{\mathcal{A}}$ be the algebra obtained from $\mathcal{A}$ by adding a unit to it. Since $\mathcal{A}$ was already assumed to be unital, then the map

$$\rho : \widetilde{\mathcal{A}} \ni (a, \lambda) \mapsto (a + \lambda \mathbf{1}, \lambda) \in \mathcal{A} \times \mathbb{C}$$

is an isomorphism, with $\mathbf{1}$ the unit of $\mathcal{A}$. Less explicitly, this isomorphism had already been used in the proof of Lemmas 2.2.4 and 3.3.5.

For $\eta \in Z^{2n+1}_\lambda(\mathcal{A})$ let us now define $\tilde{\eta} \in C^{2n+1}(\widetilde{\mathcal{A}})$ by

$$\tilde{\eta}\big((a_0, \lambda_0), \ldots, (a_{2n+1}, \lambda_{2n+1})\big) = \eta(a_0, \ldots, a_{2n+1}), \qquad \forall (a_j, \lambda_j) \in \widetilde{\mathcal{A}}.$$

By cyclicity of $\eta$, $\tilde{\eta}$ clearly belongs to $C^{2n+1}_\lambda(\widetilde{\mathcal{A}})$. Let us now check that $\mathtt{b}\tilde{\eta} = 0$, which means that $\tilde{\eta} \in Z^{2n+1}_\lambda(\widetilde{\mathcal{A}})$. Indeed, for any $(a_j, \lambda_j) \in \widetilde{\mathcal{A}}$ observe first that

$$\begin{aligned}
&\tilde{\eta}\big((a_0, \lambda_0), \ldots, (a_j, \lambda_j)(a_{j+1}, \lambda_{j+1}), \ldots, (a_{2n+2}, \lambda_{2n+2})\big) \\
&= \eta(a_0, \ldots, a_j a_{j+1}, \ldots, a_{2n+2}) + \lambda_j \, \eta(a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{2n+2}) \\
&\quad + \lambda_{j+1} \, \eta(a_0, \ldots, a_j, a_{j+2}, \ldots, a_{2n+2}).
\end{aligned}$$

As a consequence, one infers from the above equality and from the equality $\mathtt{b}\eta = 0$ that

$$\begin{aligned}
&\mathtt{b}\tilde{\eta}\big((a_0, \lambda_0), \ldots, (a_{2n+2}, \lambda_{2n+2})\big) \\
&= \lambda_0 \, \eta(a_1, \ldots, a_{2n+2}) + (-1)^{2n+2} \lambda_0 \, \eta(a_{2n+2}, a_1, \ldots, a_{2n+1}) \\
&= \lambda_0 \big[\eta(a_1, \ldots, a_{2n+2}) + (-1)^{2n+1} \eta(a_1, \ldots, a_{2n+2})\big] \\
&= 0.
\end{aligned}$$

Note that the cyclicity of $\eta$ has been used for the last equality.

With the definition of $\tilde{\eta}$ at hand let us define

$$\boldsymbol{\eta} := \tilde{\eta} \circ \operatorname{diag}\big(\rho^{-1}(\cdot, 1), \ldots, \rho^{-1}(\cdot, 1)\big) : \mathcal{A}^{2n+1} \to \mathbb{C},$$

and observe that for any $v \in \mathcal{GL}(\mathcal{A})$ the equality

$$\boldsymbol{\eta}(v^{-1}, v, \ldots, v^{-1}, v) = \eta(v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \ldots, v^{-1} - \mathbf{1}, v - \mathbf{1})$$

holds. Observe also that the newly defined cyclic cocycle $\boldsymbol{\eta}$ satisfies the equality

$$\boldsymbol{\eta}(\mathbf{1}, a_1, \ldots, a_{2n+1}) = 0.$$

Such a cocycle is called a *normalized* cocycle on $\mathcal{A}$.

Let us now show that (9.17) depends only on the cyclic cohomology class of $\eta$ in $HC^{2n+1}(\mathcal{A})$. For that purpose, let us assume that $\boldsymbol{\eta} = \mathtt{b}\psi$ for some $\psi \in C^{2n}_{\lambda}(\mathcal{A})$ with $\psi(\mathbf{1}, a_1, \ldots, a_{2n}) = 0$. Note that this normalization is indeed possible thanks to the equality $\mathtt{b}\tilde{\psi} = \widetilde{\mathtt{b}\psi}$. Then by taking the normalization and the cyclicity of $\psi$ into account, one directly infers that

$$\begin{aligned}
&[\mathtt{b}\psi](v^{-1}, v, \ldots, v^{-1}, v) \\
&= \sum_{j=0}^{2n} (-1)^j \psi(v^{-1}, \ldots, \mathbf{1}, \ldots, v) + (-1)^{2n+1} \psi(\mathbf{1}, \ldots, v) \\
&= 0,
\end{aligned}$$

as expected.

In the next statement, we consider the action of the periodicity operator, as in Lemma 9.5.2 for even cyclic cocycles. As before, without loss of generality it is sufficient to consider the case $q = 1$. Let us stress that the following statement holds for the periodicity operator introduced at the end of Section 9.2, any other convention would lead to other constants[10].

**Lemma 9.5.6.** *For any normalized $\boldsymbol{\eta} \in Z^{2n+1}_{\lambda}(\mathcal{A})$ and any $v \in \mathcal{GL}(\mathcal{A})$ one has*

$$[S\boldsymbol{\eta}](\underbrace{v^{-1}, v, \ldots, v^{-1}, v}_{2n+4 \ \text{terms}}) = (2n+2)\,[\boldsymbol{\eta}](\underbrace{v^{-1}, v, \ldots, v^{-1}, v}_{2n+2 \ \text{terms}}).$$

*Proof.* For that purpose, recall that the periodicity operator $S$ has been defined at the end of Section 9.2. Then, by using its explicit expression and the fact that $v^{-1}v = \mathbf{1} =$

[10] Be aware that the constants appearing in the literature are not consistent between different authors, or even in different papers of the same author.

$vv^{-1}$ one gets

$$\begin{aligned}
[S\boldsymbol{\eta}](\underbrace{v^{-1}, v, v^{-1} \dots, v}_{2n+4 \text{ terms}}) &= \hat{\boldsymbol{\eta}}(v^{-1} \underbrace{\mathtt{d}v\, \mathtt{d}v^{-1} \dots \mathtt{d}v}_{2n+1 \text{ terms}}) + \hat{\boldsymbol{\eta}}(v^{-1}\, \mathtt{d}v\, \mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&\quad + \sum_{i=3}^{2n+1} \hat{\boldsymbol{\eta}}(v^{-1}\, \mathtt{d}v\, \mathtt{d}v^{-1} \dots \mathtt{d}v) + \hat{\boldsymbol{\eta}}(v^{-1}\, \mathtt{d}v\, \mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&= (2n+2)\, \hat{\boldsymbol{\eta}}(v^{-1}\, \mathtt{d}v\, \mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&= (2n+2)\, \boldsymbol{\eta}(\underbrace{v^{-1}, v, v^{-1} \dots, v}_{2n+2 \text{ terms}}),
\end{aligned}$$

which leads directly to the result. □

By summing up the results obtained above, one can now state the main result for odd cocycles:

**Proposition 9.5.7.** *For any $q \in \mathbb{N}^*$ and $n \in \mathbb{N}$, the following map is bilinear*

$$K_1(\mathcal{A}) \times HC^{2n+1}(\mathcal{A}) \ni \ ([v]_1, [\eta]) \mapsto \langle [v]_1, [\eta] \rangle \ \in \mathbb{C}$$

*with*

$$\langle [v]_1, [\eta] \rangle := C_{2n+1}\, [\eta \# \mathrm{tr}](\underbrace{v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \dots, v^{-1} - \mathbf{1}, v - \mathbf{1}}_{2n+2 \ \textit{terms}})$$

*with $v \in \mathcal{GL}_q(\mathcal{A})$ and $\eta \in Z^{2n+1}_{\lambda}(\mathcal{A})$. In addition, the constants $C_{2n+1}$ can be chosen iteratively such that the following property holds :*

$$\langle [v]_1, [S\eta] \rangle = \langle [v]_1, [\eta] \rangle. \tag{9.18}$$

Part of the proof of this statement has already been provided above. For example, its has been shown that the pairing depends only on the cohomology class of $\eta$, and that the relation (9.18) holds if and only if

$$C_{2n+3}\, (2n+2) = C_{2n+1}.$$

We emphasize once more that this relation is a by-product of the choice of the normalization for the periodicity operator $S$. In addition, the value of the constant $C_1$ can still be chosen arbitrarily.

On the other hand and quite unfortunately, the proof of the independence on the choice of a representative in the $K_1$-class of $v$ is out of reach with the material introduced in these lecture notes. At first, a precise description of $K_1(\mathcal{A})$ should be provided, and more information on the Connes-Chern character are also necessary. Note also that our construction of the cyclic cohomology of $\mathcal{A}$ was based only on one boundary map, namely $\mathtt{b}$. However, a more efficient construction involving a second map $B$ and the Hochschild cohomology $H^n(\mathcal{A}, \mathcal{A}^*)$ is introduced in [Con94, Chap. III] and this approach leads more easily to some stronger results, see also [Kha13, Chap. 3].

Based on the this construction and by taking the result of Section 9.4 into account, the main result of this chapter now reads:

**Theorem 9.5.8.** *Let $\mathcal{B}$ be a unital $C^*$-algebra endowed with a $n$-trace $\eta$ defined on a dense subalgebra denoted by $\mathcal{A}$.*

*(i) If $n$ is even, there exists a map $\varphi : K_0(\mathcal{B}) \to \mathbb{C}$ such that for any $p \in \mathcal{P}_q(\mathcal{A})$ one has*

$$\varphi([p]_0) = \frac{1}{(n/2)!}[\eta\#\mathrm{tr}](p, \ldots, p), \tag{9.19}$$

*(ii) If $n$ is odd, there exists a map $\varphi : K_1(\mathcal{B}) \to \mathbb{C}$ such that for any $v \in \mathcal{GL}_q(\mathcal{A})$ one has*

$$\varphi([w(v)]_1) = C_n[\eta\#\mathrm{tr}](v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \ldots, v^{-1} - \mathbf{1}, v - \mathbf{1}), \tag{9.20}$$

*where $w(v)$ is defined by $w(v) = v|v|^{-1} \in \mathcal{U}_q(\mathcal{B})$. With our normalization for the periodicity operator $S$, the constants $C_n$ satisfy the relations $C_{n+2}(n+1) = C_n$.*

*Proof.* This statement is a direct transcription of the content of Propositions 9.5.5 and 9.5.7 once there exist isomorphisms between the $K$-groups $K_i(\mathcal{A})$ and $K_i(\mathcal{B})$. In fact, instead of $\mathcal{A}$ one has to consider the slightly larger algebra $\mathscr{B}$ introduced in Section 9.4. By Proposition 9.4.6 one only has to extend the cocycle $\eta$ to a cyclic cocycle on $\mathscr{B}$. For that purpose, it is sufficient to show that for any $\delta$-bounded sequences $\{a_{j,m}\}_m \subset \mathcal{A}$ with $a_{j,m} \to a_{j,\infty}$ the sequence $\eta(a_{0,m}, \ldots, a_{n,m})$ converges to a limit as $m \to \infty$. However, with the notations of the Lemma 9.4.9 and of its proof one has

$$\eta(a_{0,m}, \ldots, a_{n,m}) = \varphi_m(a_{0,m}, \mathbf{1}, \ldots, \mathbf{1}),$$

and it is precisely proved in Lemma 9.4.9 that this expression has a limit as $m$ goes to infinity. $\square$

# Chapter 10

# Application: Levinson's theorem

In this chapter, we briefly describe how the formalism introduced in the previous chapters leads to some index theorems in the context of scattering theory. Obviously, we shall only scratch the surface, and most the previous material is not really necessary for the example presented hereafter. However, for more involved examples this material turns out to be essential. We refer to [Ric15] for more information on the subject.

## 10.1 The ■-anisotropic algebra

In this section we briefly construct a $C^*$-algebra which will play a major role later on. This algebra has been introduced in [GI03, Sec. 3.5] for a different purpose, and we refer to this paper for the details of the construction.

In the Hilbert space $L^2(\mathbb{R})$ we consider the two canonical self-adjoint operators $X$ of multiplication by the variable, and $D = -i\frac{\mathrm{d}}{\mathrm{d}x}$ of differentiation. These operators satisfy the canonical commutation relation written formally $[iD, X] = 1$, or more precisely $e^{-isX}e^{-itD} = e^{-ist}e^{-itD}e^{-isX}$. We recall that the spectrum of both operators is $\mathbb{R}$. Then, for any functions $\varphi, \eta \in L^\infty(\mathbb{R})$, one can consider by bounded functional calculus the operators $\varphi(X)$ and $\eta(D)$ in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. And by mixing some operators $\varphi_i(X)$ and $\eta_i(D)$ for suitable functions $\varphi_i$ and $\eta_i$, we are going to produce an algebra $\mathcal{C}$ which will be useful in many applications.

Let us consider the closure in $\mathcal{B}\big(L^2(\mathbb{R})\big)$ of the $C^*$-algebra generated by elements of the form $\varphi_i(D)\eta_i(X)$, where $\varphi_i, \eta_i$ are continuous functions on $\mathbb{R}$ which have limits at $\pm\infty$. Stated differently, $\varphi_i, \eta_i$ belong to $C([-\infty, +\infty])$. Note that this algebra is clearly unital. In the sequel, we shall use the following notation:

$$\mathcal{C}_{(D,X)} := C^*\Big(\varphi_i(D)\eta_i(X) \mid \varphi_i, \eta_i \in C([-\infty, +\infty])\Big).$$

Let us also consider the $C^*$-algebra generated by $\varphi_i(D)\eta_i(X)$ with $\varphi_i, \eta_i \in C_0(\mathbb{R})$, which means that these functions are continuous and vanish at $\pm\infty$. As easily observed, this algebra is a closed ideal in $\mathcal{C}_{(D,X)}$ and is equal to the $C^*$-algebra $\mathcal{K}\big(L^2(\mathbb{R})\big)$ of compact operators in $L^2(\mathbb{R})$, see for example [GI03, Corol. 2.18].

Let us now study the quotient $C^*$-algebra $\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)$. For that purpose, we consider the square $\blacksquare := [-\infty,+\infty]\times[-\infty,+\infty]$ whose boundary $\square$ is the union of four parts: $\square = C_1\cup C_2\cup C_3\cup C_4$, with $C_1=\{-\infty\}\times[-\infty,+\infty]$, $C_2=[-\infty,+\infty]\times\{+\infty\}$, $C_3=\{+\infty\}\times[-\infty,+\infty]$ and $C_4=[-\infty,+\infty]\times\{-\infty\}$. We can also view $C(\square)$ as the subalgebra of

$$C([-\infty,+\infty])\oplus C([-\infty,+\infty])\oplus C([-\infty,+\infty])\oplus C([-\infty,+\infty]) \tag{10.1}$$

given by elements $\Gamma := (\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$ which coincide at the corresponding end points, that is, $\Gamma_1(+\infty)=\Gamma_2(-\infty)$, $\Gamma_2(+\infty)=\Gamma_3(+\infty)$, $\Gamma_3(-\infty)=\Gamma_4(+\infty)$, and $\Gamma_4(-\infty)=\Gamma_1(-\infty)$. Then $\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)$ is isomorphic to $C(\square)$, and if we denote the quotient map by

$$q:\mathcal{C}_{(D,X)}\to\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)\cong C(\square)$$

then the image $q\big(\varphi(D)\eta(X)\big)$ in (10.1) is given by $\Gamma_1=\varphi(-\infty)\eta(\cdot)$, $\Gamma_2=\varphi(\cdot)\eta(+\infty)$, $\Gamma_3=\varphi(+\infty)\eta(\cdot)$ and $\Gamma_4=\varphi(\cdot)\eta(-\infty)$. Note that this isomorphism is proved in [GI03, Thm. 3.22]. In summary, we have obtained the short exact sequence

$$0\to\mathcal{K}\big(L^2(\mathbb{R})\big)\hookrightarrow\mathcal{C}_{(D,X)}\xrightarrow{q}C(\square)\to 0 \tag{10.2}$$

with $\mathcal{K}\big(L^2(\mathbb{R})\big)$ and $\mathcal{C}_{(D,X)}$ represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$, but with $C(\square)$ which is not naturally represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. Note however that each of the four functions summing up in an element of $C(\square)$ can individually be represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$, either as a multiplication operator or as a convolution operator.

We shall now construct several isomorphic versions of these algebras. First of all, let us consider the Hilbert space $L^2(\mathbb{R}_+)$ and the action of the dilation group. More precisely, we consider the unitary group $\{U_t\}_{t\in\mathbb{R}}$ acting on any $f\in L^2(\mathbb{R}_+)$ as

$$[U_tf](x)=e^{t/2}f\big(e^tx\big),\qquad\forall x\in\mathbb{R}_+ \tag{10.3}$$

which is usually called *the unitary group of dilations*, and denote its self-adjoint generator by $A$ and call it *the generator of dilations*.

Let also $B$ be the operator of multiplication in $L^2(\mathbb{R}_+)$ by the function $-\ln$, *i.e.* $[Bf](\lambda)=-\ln(\lambda)f(\lambda)$ for any $f\in C_{\mathrm{c}}(\mathbb{R}_+)$ and $\lambda\in\mathbb{R}_+$. Note that if one sets $L$ for the self-adjoint operator of multiplication by the variable in $L^2(\mathbb{R}_+)$, *i.e.*

$$[Lf](\lambda):=\lambda f(\lambda)\qquad f\in C_{\mathrm{c}}(\mathbb{R}_+)\text{ and }\lambda\in\mathbb{R}_+, \tag{10.4}$$

then one has $B=-\ln(L)$. Now, the equality $[iB,A]=1$ holds (once suitably defined), and the relation between the pair of operators $(D,X)$ in $L^2(\mathbb{R})$ and the pair $(B,A)$ in $L^2(\mathbb{R}_+)$ is well-known and corresponds to the Mellin transform. Indeed, let $\mathscr{V}:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ be defined by $(\mathscr{V}f)(x):=e^{x/2}f(e^x)$ for $x\in\mathbb{R}$, and remark that $\mathscr{V}$ is a unitary map with adjoint $\mathscr{V}^*$ given by $(\mathscr{V}^*g)(\lambda)=\lambda^{-1/2}g(\ln\lambda)$ for $\lambda\in\mathbb{R}_+$. Then, the Mellin transform $\mathscr{M}:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ is defined by $\mathscr{M}:=\mathcal{F}\mathscr{V}$ with $\mathcal{F}$ the usual

unitary Fourier transform[1] in $L^2(\mathbb{R})$. The main property of $\mathscr{M}$ is that it diagonalizes the generator of dilations, namely, $\mathscr{M}A\mathscr{M}^* = X$. Note that one also has $\mathscr{M}B\mathscr{M}^* = D$.

Before introducing a first isomorphic algebra, observe that if $\eta \in C([-\infty,+\infty])$, then

$$\mathscr{M}^*\eta(D)\mathscr{M} = \eta(\mathscr{M}^*D\mathscr{M}) = \eta(B) = \eta\big(-\ln(L)\big) \equiv \psi(L)$$

for some $\psi \in C([0,+\infty])$. Thus, by taking these equalities into account, it is natural to define in $\mathcal{B}\big(L^2(\mathbb{R}_+)\big)$ the $C^*$-algebra

$$\mathcal{C}_{(L,A)} := C^*\Big(\psi_i(L)\eta_i(A) \mid \psi_i \in C([0,+\infty]) \text{ and } \eta_i \in C([-\infty,+\infty])\Big),$$

and clearly this algebra is isomorphic to the $C^*$-algebra $\mathcal{C}_{(D,X)}$ in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. Thus, through this isomorphism one gets again a short exact sequence

$$0 \to \mathcal{K}\big(L^2(\mathbb{R}_+)\big) \hookrightarrow \mathcal{C}_{(L,A)} \xrightarrow{q} C(\square) \to 0$$

with the square $\square$ made of the four parts $\square = B_1 \cup B_2 \cup B_3 \cup B_4$ with $B_1 = \{0\} \times [-\infty,+\infty]$, $B_2 = [0,+\infty] \times \{+\infty\}$, $B_3 = \{+\infty\} \times [-\infty,+\infty]$, and $B_4 = [0,+\infty] \times \{-\infty\}$. In addition, the algebra $C(\square)$ of continuous functions on $\square$ can be viewed as a subalgebra of

$$C\big([-\infty,+\infty]\big) \oplus C\big([0,+\infty]\big) \oplus C\big([-\infty,+\infty]\big) \oplus C\big([0,+\infty]\big) \tag{10.5}$$

given by elements $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ which coincide at the corresponding end points, that is, $\Gamma_1(+\infty) = \Gamma_2(0)$, $\Gamma_2(+\infty) = \Gamma_3(+\infty)$, $\Gamma_3(-\infty) = \Gamma_4(+\infty)$, and $\Gamma_4(0) = \Gamma_1(-\infty)$.

Finally, if one sets $\mathcal{F}_{\rm s}$ for the unitary Fourier sine transformation in $L^2(\mathbb{R}_+)$, defined for $x, k \in \mathbb{R}_+$ and any $f \in C_{\rm c}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ by

$$[\mathcal{F}_{\rm s}f](k) := (2/\pi)^{1/2} \int_0^\infty \sin(kx) f(x)\,{\rm d}x \tag{10.6}$$

then the equalities $-A = \mathcal{F}_{\rm s}^* A \mathcal{F}_{\rm s}$ and $\sqrt{H_{\rm D}} = \mathcal{F}_{\rm s}^* L \mathcal{F}_{\rm s}$ hold, where $H_{\rm D}$ corresponds to the Dirichlet Laplacian on $\mathbb{R}_+$ (see the next section for its definition). As a consequence, note that the formal equality $[i\frac{1}{2}\ln(H_{\rm D}), A] = 1$ can also be fully justified. Moreover, by using this new unitary transformation one gets that the $C^*$-subalgebra of $\mathcal{B}\big(L^2(\mathbb{R}_+)\big)$ defined by

$$\mathcal{C}_{(H_{\rm D},A)} := C^*\Big(\psi_i(H_{\rm D})\eta_i(A) \mid \psi_i \in C([0,+\infty]) \text{ and } \varphi_i \in C([-\infty,+\infty])\Big), \tag{10.7}$$

is again isomorphic to $\mathcal{C}_{(D,X)}$. In addition, the following short exact sequence takes place

$$0 \to \mathcal{K}\big(L^2(\mathbb{R}_+)\big) \hookrightarrow \mathcal{C}_{(H_{\rm D},A)} \xrightarrow{q} C(\square) \to 0, \tag{10.8}$$

and $C(\square)$ can naturally be viewed as a subalgebra of the algebra introduced in (10.5) with suitable compatibility conditions at end points.

[1]For $f \in C_{\rm c}(\mathbb{R})$ and $x \in \mathbb{R}$ we set $[\mathcal{F}f](x) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ixy} f(y)\,{\rm d}y$.

## 10.2 Elementary scattering system

In this section we introduce an example of a scattering system for which everything can be computed explicitly. It will allow us to describe more precisely the kind of results we are looking for, without having to introduce too much information on scattering theory. In fact, we shall keep the content of this section as simple as possible.

Let us start by considering the Hilbert space $L^2(\mathbb{R}_+)$ and the Dirichlet Laplacian $H_{\mathrm{D}}$ on $\mathbb{R}_+ := (0,\infty)$. More precisely, we set $H_{\mathrm{D}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the domain $\mathsf{Dom}(H_{\mathrm{D}}) = \{f \in \mathcal{H}^2(\mathbb{R}_+) \mid f(0) = 0\}$. Here $\mathcal{H}^2(\mathbb{R}_+)$ means the usual Sobolev space on $\mathbb{R}_+$ of order 2. For any $\alpha \in \mathbb{R}$, let us also consider the operator $H^\alpha$ defined by $H^\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $\mathsf{Dom}(H^\alpha) = \{f \in \mathcal{H}^2(\mathbb{R}_+) \mid f'(0) = \alpha f(0)\}$. It is well-known that if $\alpha < 0$ the operator $H^\alpha$ possesses only one eigenvalue, namely $-\alpha^2$, and the corresponding eigenspace is generated by the function $x \mapsto e^{\alpha x}$. On the other hand, for $\alpha \geq 0$ the operators $H^\alpha$ have no eigenvalue, and so does $H_{\mathrm{D}}$.

A common object of scattering theory is defined by the following formula:

$$W_\pm^\alpha := s - \lim_{t\to\pm\infty} e^{itH^\alpha} e^{-itH_{\mathrm{D}}},$$

and these limits in the strong sense are known to exist for this model. These operators are called *the wave operators*, they are isometries, and their existence allows one to study the operator $H^\alpha$ with respect to $H_{\mathrm{D}}$. Moreover, we shall provide below a very explicit formula for these operators. Let us still stress that scattering theory is a comparison theory, one always study pairs of operators.

Our first result for this model then reads, see [Ric15, Cor. 9.3] for its proof:

**Lemma 10.2.1.** *The following equalities hold:*

$$W_-^\alpha = 1 + \tfrac{1}{2}\big(1 + \tanh(\pi A) - i\cosh(\pi A)^{-1}\big)\Big[\frac{\alpha + i\sqrt{H_{\mathrm{D}}}}{\alpha - i\sqrt{H_{\mathrm{D}}}} - 1\Big],$$
$$W_+^\alpha = 1 + \tfrac{1}{2}\big(1 - \tanh(\pi A) + i\cosh(\pi A)^{-1}\big)\Big[\frac{\alpha - i\sqrt{H_{\mathrm{D}}}}{\alpha + i\sqrt{H_{\mathrm{D}}}} - 1\Big].$$

It clearly follows from these explicit formulas that the operators $W_\pm^\alpha$ belong to the algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}$ introduced in (10.7). Since these operators are also isometries with a finite dimensional co-kernel, they can be considered as lifts for their image in the quotient algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}/\mathcal{K}\big(L^2(\mathbb{R}_+)\big)$. We shall come back to this approach involving algebras in the next session, and work very explicitly for the time being.

Motivated by the above formula for $W_-^\alpha$, let us now introduce the complex function

$$\Gamma_\blacksquare^\alpha : [0,+\infty]\times[-\infty,+\infty] \ni (x,y) \mapsto 1 + \tfrac{1}{2}\big(1 + \tanh(\pi y) - i\cosh(\pi y)^{-1}\big)\Big[\frac{\alpha + i\sqrt{x}}{\alpha - i\sqrt{x}} - 1\Big].$$

Since this function is continuous on the square $\blacksquare := [0,+\infty]\times[-\infty,+\infty]$, its restriction on the boundary $\square$ of the square is also well defined and continuous. Note that this boundary is made of four parts: $\square = B_1 \cup B_2 \cup B_3 \cup B_4$ with $B_1 = \{0\}\times[-\infty,+\infty]$,

$B_2 = [0, +\infty] \times \{+\infty\}$, $B_3 = \{+\infty\} \times [-\infty, +\infty]$, and $B_4 = [0, +\infty] \times \{-\infty\}$, and that the algebra $C(\square)$ of continuous functions on $\square$ can be viewed as a subalgebra of (10.5) with the necessary compatibility conditions at the end points. With these notations, the restriction function $\Gamma^\alpha_\square := \Gamma^\alpha_\blacksquare\big|_\square$ is given for $\alpha \neq 0$ by

$$\Gamma^\alpha_\square = \Big(1, \frac{\alpha + i\sqrt{\cdot}}{\alpha - i\sqrt{\cdot}}, -\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, 1\Big) \tag{10.9}$$

and for $\alpha = 0$ by

$$\Gamma^0_\square := \Big(-\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, -1, -\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, 1\Big). \tag{10.10}$$

For simplicity, we have directly written this function in the representation provided by (10.5).

Let us now observe that the boundary $\square$ of $\blacksquare$ is homeomorphic to the circle $\mathbb{S}$. Observe in addition that the function $\Gamma^\alpha_\square$ takes its values in the unit circle $\mathbb{T}$ of $\mathbb{C}$. Then, since $\Gamma^\alpha_\square$ is a continuous function on the closed curve $\square$ and takes values in $\mathbb{T}$, its winding number $\mathrm{Wind}(\Gamma^\alpha_\square)$ is well defined and can easily be computed. So, let us compute separately the contribution $w_j(\Gamma^\alpha_\square)$ to this winding number on each component $B_j$ of $\square$. By convention, we shall turn around $\square$ clockwise, starting from the left-down corner, and the increase in the winding number is also counted clockwise. Let us stress that the contribution on $B_3$ has to be computed from $+\infty$ to $-\infty$, and the contribution on $B_4$ from $+\infty$ to 0. Without difficulty one gets:

| | $w_1(\Gamma^\alpha_\square)$ | $w_2(\Gamma^\alpha_\square)$ | $w_3(\Gamma^\alpha_\square)$ | $w_4(\Gamma^\alpha_\square)$ | $\mathrm{Wind}(\Gamma^\alpha_\square)$ |
|---|---|---|---|---|---|
| $\alpha < 0$ | 0 | $1/2$ | $1/2$ | 0 | 1 |
| $\alpha = 0$ | $-1/2$ | 0 | $1/2$ | 0 | 0 |
| $\alpha > 0$ | 0 | $-1/2$ | $1/2$ | 0 | 0 |

By comparing the last column of this table with the information on the eigenvalues of $H^\alpha$ mentioned at the beginning of the section one gets:

**Proposition 10.2.2.** *For any $\alpha \in \mathbb{R}$ the following equality holds:*

$$\mathrm{Wind}(\Gamma^\alpha_\square) = \textit{ number of eigenvalues of } H^\alpha. \tag{10.11}$$

The content of this proposition is an example of Levinson's theorem. Indeed, it relates the number of bound states of the operator $H^\alpha$ to a quantity computed on the scattering part of the system. Let us already mention that the contribution $w_2(\Gamma^\alpha_\square)$ is the only one usually considered in the literature. However, we can immediately observe that if $w_1(\Gamma^\alpha_\square)$ and $w_3(\Gamma^\alpha_\square)$ are disregarded, then no meaningful statement can be obtained.

Obviously, the above result should now be recast in a more general framework, and the algebraic background should be taken into account. Indeed, except for very specific models, it is usually not possible to compute precisely both sides of (10.11), but such an equality still holds in a much more general setting. The next section shows how $K$-theory can provide an insight on Levinson's theorem.

## 10.3 The abstract topological Levinson's theorem

Before stating the main result of this chapter, let us reformulate the content of Proposition 6.2.4.(*ii*). The key point in the next statement is that the central role is played by the partial isometry in $\mathcal{C}$ instead of the unitary element in $\mathcal{Q}$. In fact, the following statement is at the root of our topological approach of Levinson's theorem.

**Proposition 10.3.1.** *Consider the short exact sequence*

$$0 \to \mathcal{J} \hookrightarrow \mathcal{C} \xrightarrow{q} \mathcal{Q} \to 0$$

*with $\mathcal{C}$ unital. Let $W$ be a partial isometry in $M_n(\mathcal{C})$ and assume that $\Gamma := q(W)$ is a unitary element of $M_n(\mathcal{Q})$. Then $\mathbf{1}_n - W^*W$ and $\mathbf{1}_n - WW^*$ are projections in $M_n(\mathcal{J})$, and*

$$\mathrm{ind}([q(W)]_1) := \delta_1([q(W)]_1) = [\mathbf{1}_n - W^*W]_0 - [\mathbf{1}_n - WW^*]_0 \ .$$

In order to go one more step in our construction, let us add some information about some special $K$-groups, as already mentioned in Example 4.3.6 and in Example 7.4.3.

**Example 10.3.2.** *(i) Let $C(\mathbb{S})$ denote the $C^*$-algebra of continuous functions on the unit circle $\mathbb{S}$, with the $L^\infty$-norm, and let us identify this algebra with $\big\{\zeta \in C([0,2\pi]) \mid \zeta(0) = \zeta(2\pi)\big\}$, also endowed with the $L^\infty$-norm. Some unitary elements of $C(\mathbb{S})$ are provided for any $m \in \mathbb{Z}$ by the functions*

$$\zeta_m : [0,2\pi] \ni \theta \mapsto e^{-im\theta} \in \mathbb{T}.$$

*Clearly, for two different values of $m$ the functions $\zeta_m$ are not homotopic, and thus define different classes in $K_1\big(C(\mathbb{S})\big)$. With some more efforts one can show that these elements define in fact all elements of $K_1\big(C(\mathbb{S})\big)$, and indeed one has*

$$K_1\big(C(\mathbb{S})\big) \cong \mathbb{Z}.$$

*Note that this isomorphism is implemented by the winding number* $\mathrm{Wind}(\cdot)$*, which is roughly defined for any continuous function on $\mathbb{S}$ with values in $\mathbb{T}$ as the number of times this function turns around $0$ along the path from $0$ to $2\pi$. Clearly, for any $m \in \mathbb{Z}$ one has $\mathrm{Wind}(\zeta_m) = m$. More generally, if* $\det$ *denotes the determinant on $M_n(\mathbb{C})$ then the mentioned isomorphism is given by* $\mathrm{Wind} \circ \det$ *on $\mathcal{U}_n\big(C(\mathbb{S})\big)$.*

*(ii) Let $\mathcal{K}(\mathcal{H})$ denote the $C^*$-algebra of all compact operators on a infinite dimensional and separable Hilbert space $\mathcal{H}$. For any $n$ one can consider the orthogonal projections on subspaces of dimension $n$ of $\mathcal{H}$, and these finite dimensional projections belong to $\mathcal{K}(\mathcal{H})$. It is then not too difficult to show that two projections of the same dimension are Murray-von Neumann equivalent, while projections corresponding to two different values of $n$ are not. With some more efforts, one shows that the dimension of these projections plays the crucial role for the definition of $K_0\big(\mathcal{K}(\mathcal{H})\big)$, and one has again*

$$K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}.$$

*In this case, the isomorphism is provided by the usual trace* Tr *on finite dimensional projections, and by the tensor product of this trace with the trace* tr *on* $M_n(\mathbb{C})$*. More precisely, on any element of* $\mathcal{P}_n\big(\mathcal{K}(\mathcal{H})\big)$ *the mentioned isomorphism is provided by* $\mathrm{Tr} \circ \mathrm{tr}$.

Let us now add the different pieces of information we have presented so far, and get an abstract version of our Levinson's theorem. For that purpose, we consider an arbitrary separable Hilbert space $\mathcal{H}$ and a unital $C^*$-subalgebra $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$ which contains the ideal of $\mathcal{K}(\mathcal{H})$ of compact operators. We can thus look at the short exact sequence of $C^*$-algebras

$$0 \to \mathcal{K}(\mathcal{H}) \hookrightarrow \mathcal{C} \xrightarrow{q} \mathcal{C}/\mathcal{K}(\mathcal{H}) \to 0.$$

Let us assume in addition that $\mathcal{C}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C(\mathbb{S})$. Then, if we take the results presented in the previous example into account, one infers that

$$\mathbb{Z} \cong K_1\big(C(\mathbb{S})\big) \xrightarrow{\text{ind}} K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$$

with the first isomorphism realized by the winding number and the second isomorphism realized by the trace. As a consequence, one infers from this together with Proposition 10.3.1 that there exists $\mathfrak{n} \in \mathbb{Z}$ such that for any partial isometry $W \in \mathcal{C}$ with unitary $\Gamma := q(W) \in C(\mathbb{S})$ the following equality holds:

$$\mathrm{Wind}(\Gamma) = \mathfrak{n}\mathrm{Tr}\big([\mathbf{1} - W^*W] - [\mathbf{1} - WW^*]\big). \tag{10.12}$$

We emphasize once again that the interest in this equality is that the left hand side is independent of the choice of any special representative in $[\Gamma]_1$. On the other hand, in the context of scattering theory the r.h.s. of (10.12) is well understood, see the next statement. Let us also mention that the number $\mathfrak{n}$ depends on the choice of the extension of $\mathcal{K}(\mathcal{H})$ by $C(\mathbb{S})$, see [W-O93, Chap. 3.2], but also on the convention chosen for the computation of the winding number.

If we summarize all this in a single statement, one gets:

**Theorem 10.3.3** (Abstract topological Levinson's theorem)**.** *Let* $\mathcal{H}$ *be a separable Hilbert space, and let* $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ *be a unital* $C^*$*-algebra such that* $\mathcal{K}(\mathcal{H}) \subset \mathcal{C}$ *and* $\mathcal{C}/\mathcal{K}(\mathcal{H}) \cong C(\mathbb{S})$ *(with quotient morphism denoted by* $q$*). Then there exists* $\mathfrak{n} \in \mathbb{Z}$ *such that for any partial isometry* $W \in \mathcal{C}$ *with unitary* $\Gamma := q(W) \in C(\mathbb{S})$ *the following equality holds:*

$$\mathrm{Wind}(\Gamma) = \mathfrak{n}\mathrm{Tr}\big([\mathbf{1} - W^*W] - [\mathbf{1} - WW^*]\big). \tag{10.13}$$

*In particular if* $W = W_- = s - \lim_{t\to-\infty} e^{itH}e^{-itH_0}$ *for some suitable scattering pair* $(H, H_0)$*, then the previous equality reads*

$$\mathrm{Wind}\big(q(W_-)\big) = -\mathfrak{n}\big(\textit{number of eigenvalues of } H - \textit{number of eigenvalues of } H_0\big).$$

Note that in applications, the factor $\mathfrak{n}$ is determined by computing both sides of the equality on an explicit example.

Let us finally show that the example presented in Section 10.2 can be recast in the previous framework. We consider the Hilbert space $L^2(\mathbb{R}_+)$ and the unital $C^*$-algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}$ introduced in (10.7). As already mentioned, the wave operator $W_-^\alpha$ is an isometry which clearly belongs to the $C^*$-algebra $\mathcal{C}_{(H_{\mathrm{D}},A)} \subset \mathcal{B}\big(L^2(\mathbb{R}_+)\big)$. In addition, the image of $W_-^\alpha$ in the quotient algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}/\mathcal{K}\big(L^2(\mathbb{R}_+)\big) \cong C(\square)$ is precisely the function $\Gamma_\square^\alpha$, defined in (10.9) for $\alpha \neq 0$ and in (10.10) for $\alpha = 0$, which are unitary elements of $C(\square)$. Finally, since $C(\square)$ and $C(\mathbb{S})$ are clearly isomorphic, the winding number $\mathrm{Wind}(\Gamma_\square^\alpha)$ of $\Gamma_\square^\alpha$ can be computed, and in fact this has been performed and recorded in the table of Section 10.2.

If one sets $E_p(H^\alpha)$ for the orthogonal projection on the subspace generated by the bound states of the operator $H^\alpha$, then one has

$$\mathrm{Tr}\big([\mathbf{1} - (W_-^\alpha)^* W_-^\alpha] - [\mathbf{1} - W_-^\alpha (W_-^\alpha)^*]\big) = -\mathrm{Tr}\big(E_p(H^\alpha)\big) = \begin{cases} -1 & \text{if } \alpha < 0\,, \\ 0 & \text{if } \alpha \geq 0\,. \end{cases} \tag{10.14}$$

Thus, this example fits in the framework of Theorem 10.3.3, and in addition both sides of (10.13) have been computed explicitly. By comparing (10.14) with the results obtained for $\mathrm{Wind}(\Gamma_\square^\alpha)$, one gets that the factor $\mathfrak{n}$ mentioned in (10.13) is equal to $-1$ for these algebras. Finally, since $E_p(H^\alpha)$ is related to the point spectrum of $H^\alpha$, the content of Proposition 10.2.2 can be rewritten as

$$\mathrm{Wind}(\Gamma_\square^\alpha) = \sharp\sigma_{\mathrm{p}}(H^\alpha).$$

This equality corresponds to a topological version of Levinson's theorem for the elementary model. Obviously, this result was already obtained in Section 10.2 and all the above framework was not necessary for its derivation. However, we have now in our hands a very robust framework which can be applied to several other situations, see [Ric15] and the references therein.

**Remark 10.3.4.** *As a concluding remark, let us mention how the algebraic framework could still be extended. For that purpose, consider a short exact sequence*

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{C} \longrightarrow \mathcal{Q} \longrightarrow 0$$

*and the corresponding index map* $\mathrm{ind} : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$. *Assume that $\eta$ is an even $n$-trace on $\mathcal{J}$ which can be paired with $K_0(\mathcal{J})$, see Theorem 9.5.8. Then one can wonder if there exists a map on higher traces which is dual to the index map, i.e. a map $\#$ which assigns to an even trace $\eta$ an odd trace $\#\eta$ such that the equality*

$$\big\langle [\mathrm{ind}(\Gamma)]_0, [\eta] \big\rangle = \big\langle [\Gamma]_1, [\#\eta] \big\rangle \tag{10.15}$$

*holds, for any $\Gamma \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ ? Except for some special cases (like in Theorem 10.3.3 for a 0-trace and a 1-trace), the answer to this question is apparently not known.*

# Bibliography


[Con94] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[Con85] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Etudes Sci. Publ. Math. **62** (1985), 257–360.

[Con86] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*, in Geometric methods in operator algebras (Kyoto, 1983), 52–144, Pitman Res. Notes Math. Ser. 123, Longman Sci. Tech., Harlow, 1986.

[CMR07] J. Cuntz, R. Meyer, J. Rosenberg, *Topological and bivariant K-theory*, Oberwolfach Seminars, 36. Birkhäuser Verlag, Basel, 2007.

[GI03] V. Georgescu, A. Iftimovici, *$C^*$-algebras of quantum Hamiltonians*, in Operator Algebras and Mathematical Physics, Conference Proceedings: Constanţa (Romania) July 2001, 123–167, Theta Foundation, 2003.

[GVF01] J. Gracia-Bondía, J Várilly, H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts, Birkhäuser Boston, Inc., Boston, MA, 2001.

[Kha13] M. Khalkhali, *Basic noncommutative geometry*, second edition, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2013.

[KS04] J. Kellendonk, H. Schulz-Baldes, *Boundary maps for $C^*$-crossed products with $\mathbb{R}$ with an application to the quantum Hall effect*, Comm. Math. Phys. **249** no. 3 (2004), 611–637.

[MN08] H. Moriyoshi, T. Natsume, *Operator algebras and geometry*, Translations of Mathematical Monographs **237**, American Mathematical Society, Providence, RI, 2008.

[Mur90] G.J. Murphy, *$C^*$-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[Ped89] G. Pedersen, *Analysis now*, Graduate Texts in Mathematics, 118, Springer-Verlag, New York, 1989.

[Ric15] S. Richard, *Levinson's theorem: an index theorem in scattering theory*, to appear in the Proceedings of the Conference Spectral Theory and Mathematical Physics, Santiago 2014, Operator Theory Advances and Application, Birkhäuser.

[RLL00] M. Rørdam, F. Larsen, N. Laustsen, *An introduction to K-theory for $C^*$-algebras*, London Mathematical Society Student Texts **49**, Cambridge University Press, Cambridge, 2000.

[Yos65] K. Yosida, *Functional analysis*, Die Grundlehren der Mathematischen Wissenschaften **123**, Academic Press, Inc., New York; Springer-Verlag, Berlin 1965.

[W-O93] N.E. Wegge-Olsen, *K-theory and $C^*$-algebras, a friendly approach*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.