Special Math Lecture
Graph Theory

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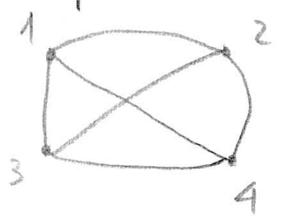
1, <u>Proposition</u>: For any even number n greater or equal to 4, there exists a 3-regular graph of order n.

<u>proof</u>: Using induction

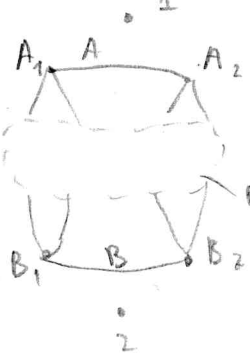
- Base step: $n = 4$



+ Graph has 4 vertices 1, 2, 3, 4.
  $V = \{1, 2, 3, 4\}$; and 6 edges as in the figure
+ Observe that:
  . $\deg(1) = 3$
  . $\deg(2) = 3$
  . $\deg(3) = 3$
  . $\deg(4) = 3$
  (Each vertex has 3 edges)

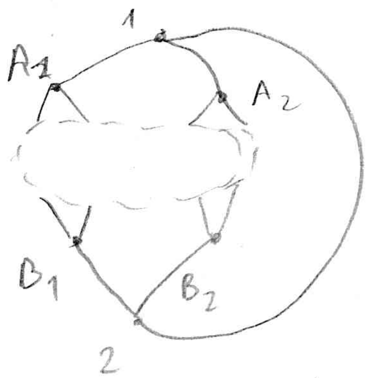
Thus, $\deg(x) = 3$ for all $x \in V$.
Therefore, this graph is a 3-regular graph of order 4.

- Induction step: Assume there exists a 3-regular graph of order n ($n = 2k$, $k \in \mathbb{N}$, $k > 1$)



We add 2 more vertices to the graph by the following rules:

- Choose 2 pairs of vertices that each pair of vertices is connected by one edge, namely:
  + edge A connecting A_1 and A_2
  + edge B connecting B_1 and B_2
- Then delete those 2 edges, connect each pair of vertices to a new vertex just added.
  (A_1 and A_2 are connected to 1
  B_1 and B_2 are connected to 2)
- Connect the 2 new vertices (1 and 2)



*Check: +, A_1 loses one edge to A_2 and is added a new one to 1. Therefore, $\deg(A_1) = 3$ (unchanged)
* Similarly, $\deg(A_2) = \deg(B_1) = \deg(B_2) = 3$ (unchanged)

For vertex 1, it is connected to A_1, A_2 and 2 by 3 edges $\Rightarrow \deg(1) = 3$
Similarly, $\deg(2) = 3$
Therefore, this new graph is a 3-regular graph of order n (added 2 more vertices)
The proposition is true for $n+2$ if it is true for n (n is even ($n \geq 4$))

By Induction, the proposition is true for all even n, greater or equal to 4.

2, Proposition : Every tree of n-vertices contains exactly n-1 edges.

Proof : Using

* For n=2, we uniquely define a tree with 1 edge. If there was more than one edge, there would create a cycle which it is not a tree.

* Assume it's true for a tree of n-vertices, then there are exactly n-1 edges.

Consider a tree of n+1 vertices. By previous proposition, this tree contains at least 2 leaves.

If we delete 1 edge of the tree of that leave, since the graph is simple, we can isolate one vertex.

Therefore, we receive a n-vertices tree and an isolated vertex.

If the initial true has M edges, then the received n-vertices tree has M-1 edges.

On the other hand, by the assumption, the n-vertices tree has exactly n-1 edges.

Thus, $n-1 = M-1$ or $n = M$

Therefore, the initial n+1 vertices tree has n edges.

The proposition is true for n+1 vertices tree if it is true for n-vertices tree.

By induction, every tree of n-vertices contains exactly n-1 edges.