Chapter 3 Zhengliang Zhu

3.11

(a)

$$\lim_{\substack{N\to+\infty \\ M\to+\infty \\ \frac{M}{N}\to p}} P(X=x\mid N,M,k) = \lim_{\substack{N\to+\infty \\ M\to+\infty \\ \frac{M}{N}\to p}} \frac{C_M^x\, C_{N-M}^{k-x}}{C_N^k} =$$

$$= \lim_{\substack{N\to+\infty \\ M\to+\infty \\ \frac{M}{N}\to p}} \frac{\frac{M!}{x!(M-x)!}\cdot\frac{(N-M)!}{(k-x)!(N-M-k+x)!}}{\frac{N!}{k!(N-k)!}} = \lim_{\sim} \frac{M!(N-M)!\,k!\,(N-k)!}{N!\,x!\,(M-x)!\,(k-x)!\,(N-M-k+x)!}$$

Stirly formula $\lim_{n\to\infty} n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$= \lim_{\sim} \binom{k}{x} \frac{\sqrt{\cancel{2\pi} M}\left(\frac{M}{e}\right)^M\cdot\sqrt{\cancel{2\pi}(N-M)}\cdot\left(\frac{N-M}{e}\right)^{N-M}\sqrt{\cancel{2\pi}(N-k)}\cdot\left(\frac{N-k}{e}\right)^{N-k}}{\sqrt{\cancel{2\pi} N}\left(\frac{N}{e}\right)^N\sqrt{\cancel{2\pi}(M-x)}\left(\frac{M-x}{e}\right)^{M-x}\cdot\sqrt{\cancel{2\pi}(N-M-k+x)}\cdot\left(\frac{N-M-k+x}{e}\right)^{N-M-k+x}}$$

$$= \lim_{\sim} \binom{k}{x} \frac{M^{M+\frac{1}{2}}\,\cancel{e^{-M}}\cdot(N-M)^{N-M+\frac{1}{2}}\,\cancel{e^{M-N}}\cdot(N-k)^{N-k+\frac{1}{2}}\cdot\cancel{e^{k-N}}}{N^{N+\frac{1}{2}}\cdot\cancel{e^{-N}}\cdot(M-x)^{M-x+\frac{1}{2}}\cdot\cancel{e^{x-M}}\,(N-M-k+x)^{N-M-k+x+\frac{1}{2}}\cancel{e^{k+M-N-x}}}$$

$$= \lim_{\sim} \binom{k}{x} \frac{M^{M+\frac{1}{2}}\,(N-M)^{N-M+\frac{1}{2}}\,(N-k)^{N-k+\frac{1}{2}}}{N^{N+\frac{1}{2}}\,(M-x)^{M-x+\frac{1}{2}}\,(N-M-k+x)^{N-M-k+x+\frac{1}{2}}}$$

$$= \lim_{\sim} \binom{k}{x} \frac{M^x}{N^k}\cdot\frac{\left(1-\frac{k}{N}\right)^{N-k+\frac{1}{2}}}{\left(1-\frac{x}{M}\right)^{M-x+\frac{1}{2}}}\cdot(N-M)^{k-x}\left(1-\frac{k-x}{N-M}\right)^{-(N-M-k+x+\frac{1}{2})}$$

$$\left(\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x = e \qquad \lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^k = 1\right)$$

$$= \lim_{M\to pN} \binom{k}{x} \frac{p^x \cancel{N^x}}{\cancel{N^k}}\cdot(1-p)^{k-x}\cdot\cancel{N^{k-x}}\cdot\frac{e^{\cancel{k}}}{e^{\cancel{x}}}\,e^{\cancel{x-k}}$$

$$= \binom{k}{x} p^x (1-p)^{k-x}$$

(b) $\frac{M}{N} \to \frac{\lambda}{k}$

$$\lim_{\sim} \binom{k}{x} \cdot p^x (1-p)^{k-x} = \lim_{\sim} \frac{k!}{x!(k-x)!} \left(\frac{\lambda}{k}\right)^x \cdot \left(1+\frac{-\lambda}{k}\right)^{k-x}$$

$$= \lim_{\sim} \frac{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k}{x!\sqrt{2\pi(k-x)}\left(\frac{k-x}{e}\right)^{k-x}} \cdot \frac{\lambda^x}{k^x} \cdot e^{-\lambda}$$

$$= \lim_{\sim} \frac{k^{k+\frac{1}{2}}}{(k-x)^{k-x+\frac{1}{2}} \cdot e^x k^x} \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= \lim_{\sim} \frac{1}{\left(1+\frac{-x}{k}\right)^{k-x+\frac{1}{2}} \cdot e^x} \cdot \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= \lim_{\sim} \frac{1}{e^{-x} \cdot e^x} \cdot \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= \lim_{\substack{M\to\infty \\ N\to\infty}} \frac{\lambda^x}{x!} e^{-\lambda} \longrightarrow \frac{e^{-\lambda}\lambda^x}{x!}$$

$\frac{kM}{N} \to \lambda$

$k \to \infty$

(c) $$\lim_{\sim} \frac{M!(N-M)!k!(N-k)!}{N!x!(M-x)!(k-x)!(N-M-k+x)!}$$

$$= \lim_{\sim} \frac{1}{x!} \frac{M^{M+\frac{1}{2}} e^{\cancel{-M}} (N-M)^{N-M+\frac{1}{2}} e^{\cancel{M-N}} k^{k+\frac{1}{2}} e^{\cancel{-k}} (N-k)^{N-k+\frac{1}{2}} e^{\cancel{k-N}}}{N^{N+\frac{1}{2}} e^{\cancel{-N}} (M-x)^{M-x+\frac{1}{2}} e^{\cancel{x-M}} (k-x)^{k-x+\frac{1}{2}} e^{x-k} (N-M-k+x)^{N-M-k+x+\frac{1}{2}} e^{-(\cancel{N}-\cancel{M}-\cancel{k}+\cancel{x})}}$$

$$= \lim_{\sim} \frac{1}{x! e^{x}} \cdot \frac{M^x}{N^{N+\frac{1}{2}}} \frac{(N-M)^{N-M+\frac{1}{2}} k^{k+\frac{1}{2}} (N-k)^{N-k+\frac{1}{2}}}{(k-x)^{k-x+\frac{1}{2}} (N-M-k+x)^{N-M-k+x+\frac{1}{2}}}$$

$kM \to \lambda N$

$$= \lim_{\sim} \frac{1}{x!} \cdot \frac{M^x}{N^k} \cdot \frac{(N-M)^{k-x}}{e^{-x}} \cdot k^x \frac{e^{-k}}{e^{-k+x}}$$

$$= \lim_{\sim} \frac{1}{x!} \frac{(kM)^x}{N^k} \frac{(N-M)^k}{(N-M)^x} = \lim_{\sim} \frac{1}{x!} \frac{\lambda^x}{\left(1-\frac{M}{N}\right)^x} \cdot \left(1-\frac{M}{N}\right)^k = \lim_{\sim} \frac{1}{x!} \lambda^x \cdot \left(1-\frac{\lambda}{k}\right)^k$$

$$= \frac{e^{-\lambda}\lambda^x}{x!}$$

Chapter 7 Zhengliang Zhu

7.1

$L(\theta|x) = p(x|\theta)$

when $x=0$, $L(\theta|0) = p(0|\theta) = f(0|\theta)$

$L(1|0) = \frac{1}{3}$, $L(2|0) = \frac{1}{4}$, $L(3|0) = 0$

$\therefore$ when $x=0$, $\hat{\theta} = 1$

the same as others

$x=1$, $\hat{\theta} = 1$

$x=2$, $\hat{\theta} = 2$ or 3

$x=3$, $\hat{\theta} = 3$

$x=4$, $\hat{\theta} = 3$

7.2 Gamma (α, β)

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$$L(\beta|x,\alpha) = \prod_{i=1}^{n} f(x_i|\alpha,\beta) = \frac{1}{\Gamma^n(\alpha)\,\beta^{\alpha n}} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\frac{1}{\beta}\sum_{i=1}^{n} x_i}$$

$$\ln L = -n\ln\Gamma(\alpha) - \alpha n \ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$

$$* \quad \frac{\partial \ln L}{\partial \beta} = -\alpha n \frac{1}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0$$

$$\Leftrightarrow \hat{\beta} = \frac{1}{\alpha n}\sum_{i=1}^{n} x_i$$

b)

$$L(\alpha,\beta|X) = \frac{1}{\Gamma(\alpha)^n \beta^{\alpha n}} \left[\prod_{i=1}^n x_i\right]^{\alpha-1} e^{-\frac{1}{\beta}\sum_{i=1}^n x_i}$$

let $\beta = \frac{1}{\alpha n}\sum_{i=1}^n x_i$

$$L(\alpha,\beta|X) = L(\alpha|x,\hat{\beta}) = \frac{1}{\Gamma(\alpha)^n \left(\frac{1}{\alpha n}\sum_{i=1}^n x_i\right)^{\alpha n}} \left[\prod_{i=1}^n x_i\right]^{\alpha-1} e^{-\alpha n}$$

~ too complex

7.3

we need to find $\hat{\theta}$ that satisfy $\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial^2 L}{\partial \theta^2} < 0$

the $L(\hat{\theta}|x)$ is the maximum

So we need to prove

$$\frac{\partial L}{\partial \hat{\theta}}\Big|_{\hat{\theta}} = 0 \Leftrightarrow \frac{\partial \ln L}{\partial \theta}\Big|_{\hat{\theta}} = 0$$

$$\frac{\partial^2 L}{\partial \theta^2}\Big|_{\hat{\theta}} < 0 \Leftrightarrow \frac{\partial^2 \ln L}{\partial \theta^2}\Big|_{\hat{\theta}} < 0$$

$) \frac{\partial \ln L}{\partial \theta} = \frac{1}{L}\frac{\partial L}{\partial \theta}$, so if $\frac{\partial L}{\partial \theta}\Big|_{\hat{\theta}} = 0 \Leftrightarrow \frac{1}{L}\frac{\partial L}{\partial \theta}\Big|_{\hat{\theta}} = 0$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{1}{L}\frac{\partial L}{\partial \theta}\right) = \frac{1}{L}\cdot\frac{\partial^2 L}{\partial \theta^2} - \frac{1}{L^2}\frac{\partial L}{\partial \theta}$$

if $\frac{\partial L}{\partial \theta}\Big|_{\hat{\theta}} = 0$, and $\frac{\partial^2 L}{\partial \theta^2}\Big|_{\hat{\theta}} < 0 \Leftrightarrow \frac{\partial^2 \ln L}{\partial \theta^2} = \frac{1}{L}\frac{\partial^2 L}{\partial \theta^2}\Big|_{\hat{\theta}} < 0$

so they are equivalent.

or:

7.4 we have $\theta \geq 0$

$$L(\theta|x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{(-\frac{1}{2})\sum_{i=1}^{n}(x_i-\theta)^2}$$

if $\bar{x} \geq 0$, and $\frac{dL}{d\theta} = 0 \iff \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i \geq 0$

if $\bar{x} < 0$, $\frac{dL}{d\theta} = L \cdot \left(-\sum_{i=1}^{n} (\theta - x_i)\right) = L\sum_{i=1}^{n}(x_i - \theta)$

because $\bar{x} < 0$, $\theta \geq 0$

so $\frac{dL}{d\theta} < 0$, so it decreases in $[0, +\infty)$

so $\hat{\theta} = 0$.

~~7.7~~

$\theta = 0$

$$L(0|x) = \prod_{i=1}^{n} f(x_i|0) = \begin{cases} 1 & 0 < x_i < 1 \\ 0 & \text{others} \end{cases}$$

$\theta = 1$

$$L(1|x) = \prod_{i=1}^{n} f(x_i|1) = \begin{cases} \prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} & 0 < x_i < 1 \\ 0 & \text{others} \end{cases}$$

when $0 < x_i < 1$ and $\prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} > 1$, $\hat{\theta} = 1$

when $0 < x_i < 1$ and $\prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} < 1$, $\hat{\theta} = 0$

when $x_i \leq 0$, or $x_i \geq 1$ or $\prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} = 1$, $\hat{\theta} = 0$ or 1

7.8

1). $Bias_{\sigma^2}(S^2) = E_{\sigma^2}(S^2) - \sigma^2 = 0$

sample variance S^2 is unbiased estimator, so here we need X^2

$E(X^2) = E(X-0)^2 = \underset{=\sigma^2}{Var(X)}$, then $E_{\sigma^2}(X^2) - \sigma^2 = 0$, $bias_{\sigma^2}(X^2) = 0$ X^2 is unbiased estimator of σ^2.

2)

$$L(\sigma|x) = f(x|\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(-\ln\sqrt{2\pi} - \ln\sigma - \frac{x^2}{2\sigma^2}\right) = \frac{1}{-\sigma} - \frac{x^2}{2}\left(-\frac{2}{\sigma^3}\right) = 0$$ Let

$$\Leftrightarrow \frac{1}{\sigma} = \frac{x^2}{\sigma^3} \Leftrightarrow \sigma^2 = X^2 \Leftrightarrow \hat{\sigma} = |X|$$

nd. 2

$$\frac{\partial^2 \ln L}{\partial \sigma^2}\bigg|_{\hat{\sigma}} = \frac{1}{\sigma^2} - x^2 \cdot \frac{3}{\sigma^4}\bigg|_{\hat{\sigma}} = \frac{2}{-X^2} < 0$$

so $\hat{\sigma} = |X|$ is MLE

3. $f_X(x|\sigma)$

$$\hat{\mu}_j = E(X^j) = \frac{1}{n}\sum_{i=1}^{n} x_i^j$$

then $E(X) = \mu = 0$

$$E(X^2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = X^2$$

$\therefore$ $\hat{\mu}_2 = E(X^2) = X^2 = \sigma^2$

then $\hat{\sigma} = |X|$

7.11

a) $L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^{n} \prod_{i=1}^{n} x_i^{(\theta-1)}$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(n\ln\theta + \sum_{i=1}^{n} (\theta-1)\ln x_i \right) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i = 0$$

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \quad \text{so } \hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

7.12

a) $L(\theta|x) = \prod_{i=1}^{n} \theta^{xn} (1-\theta)^{(1-x)n}$

$$\frac{\partial \ln L(\theta|x)}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\sum x_i n \ln\theta + \sum (1-x_i) n \ln(1-\theta) \right) = \frac{\sum x_i n}{\theta} - \frac{\sum (1-x_i) n}{1-\theta} = 0$$

$\Leftrightarrow \hat{\theta} = \bar{x}$

$$\frac{\partial^2 \ln L}{\partial \theta} = -\frac{\sum_{i=1}^{n} x_i n}{\theta^2} - \frac{\sum_{i=1}^{n} (1-x_i) n}{(1-\theta)^2} < 0 \quad \text{so } \hat{\theta} = \bar{x}$$

and $0 \leq \theta \leq \frac{1}{2}$

so if $\bar{x} > \frac{1}{2}$ then $\hat{\theta} = \frac{1}{2}$

so $\hat{\theta} = \min\{\frac{1}{2}, \bar{x}\}$.

method of moments:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X} = \theta \Leftrightarrow \hat{\theta} = \bar{X}.$$

7.15

a)

$$L(\mu,\lambda|x) = \prod_{i=1}^{n} f(x_i|\mu,\lambda) = \left(\frac{\lambda^n}{\prod_{i=1}^{n} 2\pi x_i^3}\right)^{\frac{1}{2}} \exp\left\{\sum_{i=1}^{n} \frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right)$$

$$\underline{\ln L} = \frac{n}{2}\ln\lambda - \frac{1}{2}\sum_{i=1}^{n}\left(\ln 2\pi + 3\ln x_i\right) + \sum_{i=1}^{n}\frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}$$

$$\frac{\partial \ln L}{\partial \mu} = \sum_{i=1}^{n} -\frac{\lambda}{\cancel{2x_i}} 2 \cdot \frac{x_i-\mu}{\mu} \cdot -\frac{x_i}{\mu^2} = 0 \quad \Longleftrightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

$$\left.\frac{\partial^2 \ln L}{\partial \mu^2}\right|_{\hat{\mu}} = \sum_{i=1}^{n}\left(-\frac{3\lambda x_i}{\mu^4} + \frac{2\lambda}{\mu^3}\right)\Big|_{\hat{\mu}} = -\frac{n\lambda}{\overline{x}^3} < 0$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{2}\frac{1}{\lambda} + \sum_{i=1}^{n}\frac{-(x_i-\mu)^2}{2\mu^2 x_i} = 0$$

$$\Longleftrightarrow \hat{\lambda} = \frac{n}{2} \cdot \frac{1}{\sum_{i=1}^{n}\frac{x_i^2 - 2\mu x_i + \mu^2}{2\mu^2 x_i}} = \frac{n}{2} \cdot \frac{1}{\sum_{i=1}^{n}\left(\frac{x_i}{2\mu^2} - \frac{1}{\mu} + \frac{1}{2x_i}\right)}$$

μ substitued by $\hat{\mu} = \overline{x}$

$$= \frac{n}{2} \cdot \frac{1}{\frac{\sum_{i=1}^{n} x_i}{2\overline{x}^2} - \frac{n}{\mu} + \sum_{i=1}^{n}\frac{1}{2x_i}} = n \cdot \frac{1}{2\left(\frac{n\overline{x}}{2\overline{x}^2} - \frac{n}{\overline{x}} + \sum_{i=1}^{n}\frac{1}{2x_i}\right)}$$

$$= \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i} - \frac{n}{\overline{x}}} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\overline{x}}\right)} = \hat{\lambda}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{1}{\lambda^2}\frac{n}{2} < 0 \quad \sim$$

(4).

7.19. $t_i \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t_i}{2\sigma^2}}$

(a) $\theta = (\beta, \sigma^2)$

$$L(\theta|Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2}$$

$$= \frac{1}{(\sqrt{2\pi})^n} \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 + \beta^2 x_i^2 - 2\beta x_i y_i)}$$

so $\left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_i Y_i\right)$ is sufficient statistic.

b). $$\frac{\partial \ln L}{\partial \beta} = \frac{\partial}{\partial \beta}\left(-\frac{n}{2}\ln 2\pi - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right)$$

$$= \frac{1}{-2\sigma^2}\sum_{i=1}^{n} 2(y_i - \beta x_i)(-x_i) = \frac{1}{-\sigma^2}\left(\sum_{i=1}^{n} -x_i y_i + \sum_{i=1}^{n} \beta x_i^2\right)$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2\cdot(-x_i)(-x_i) = -\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 < 0.$$

$$E(\hat{\beta}) - \beta = \text{bias}_\beta(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right) - \beta = \frac{\sum_{i=1}^{n} x_i E(Y_i)}{\sum_{i=1}^{n} x_i^2} - \beta = \beta - \beta = 0$$

and $E(Y_i) = E(\beta x_i + \epsilon_i) = \beta E(x_i) + \mu = \beta \cdot x_i + 0 = \beta x_i$

so $\hat{\beta}$ is an unbiased estimator of β.