2.17

a)

$$\frac{\int_0^m f(x)\,dx}{\int_0^1 f(x)\,dx} = \frac{1}{2} \Rightarrow m = \sqrt[3]{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}}$$

b)

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2} \Rightarrow \left.\frac{\arctan x}{\pi}\right|_{-\infty}^{m} = \frac{1}{2} \Rightarrow m = 0$$

2.18

we have $\int_{-\infty}^{m} f(x) = \frac{1}{2}$

$$E(|x-a|) - E(|x-m|)$$

$$= \int_{-\infty}^{a}(a-x)f(x)dx + \int_{a}^{\infty}(x-a)f(x)dx - \int_{-\infty}^{m}(m-x)f(x)dx - \int_{m}^{\infty}(x-m)f(x)dx$$

$$= \int_{-\infty}^{a}(af(x) - xf(x))dx - \int_{-\infty}^{m}(mf(x) - xf(x))dx + \int_{a}^{\infty}(xf(x) - af(x))dx - \int_{m}^{\infty}(xf(x) - mf(x))dx$$

$$= \int_{-\infty}^{a} af(x) + \int_{a}^{m} xf(x)dx - \frac{1}{2}m + \int_{a}^{m} xf(x)dx + \frac{1}{2}m - a\int_{a}^{\infty} f(x)dx$$

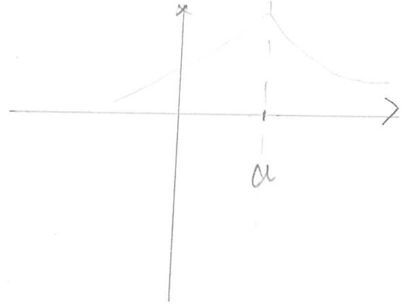
$$= 2\int_{a}^{m} xf(x)dx + \int_{-\infty}^{m} a\cancel{f(x)}dx + \int_{m}^{a} af(x)dx - \int_{a}^{m} af(x)dx - \int_{m}^{\infty} \cancel{af(x)}dx$$

$$= 2\int_{a}^{m}(x-a)f(x)dx$$

① if $a>m$ or $a<m$, $2\int_a^m (x-a)f(x)dx > 0$ ② $a=m$. $2\int_a^m (x-a)f(x)dx = 0$

so $E(|x-a|) \geq E(|x-m|)$

2.28



(a)

$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx =$

$\int_{-\infty}^{\infty} (x-a) f(x)\,dx + a\int_{-\infty}^{\infty} f(x)\,dx$

$= a$

$\mu_2 = E((x-\mu)^2) \neq 0$

$\mu_3 = E\left((x-\mu)^3\right) = \int_{-\infty}^{\infty} (x-a)^3 f(x)\,dx = 0$

because it's an odd function symmetric about $x=a$.

So $\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = 0$

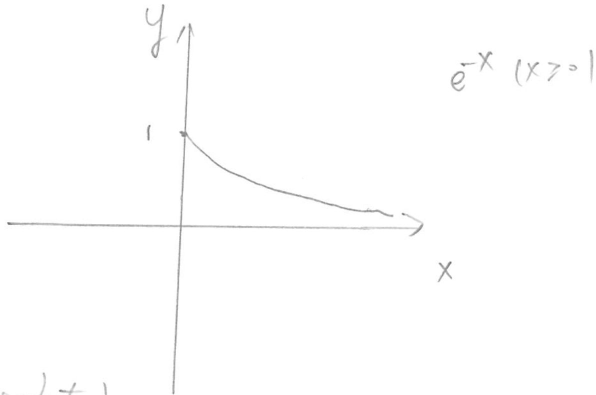
(b)

$\mu = E(x) = \int_0^{\infty} x e^{-x}\,dx = \Gamma(2) = 1$

$\mu_2 = E\left[(x-1)^2\right] = \int_0^{\infty} (x^2-2x+1)e^{-x}\,dx = \Gamma(3) - 2\Gamma(2) + \Gamma(1) = 2! - 2 + 1 = 1$

$\mu_3 = E\left[(x-1)^3\right] = \int_0^{\infty} (x^3+3x-3x^2-1)e^{-x}\,dx = \Gamma(4) - 3\Gamma(3) + 3\Gamma(2) - \Gamma(1) = 6-6+3-1$
$= 2$

$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = 2 > 0$

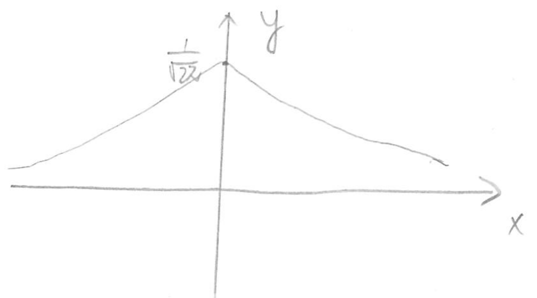


so it's skewed to the right.

(biased / distorted to the right)

(C)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$



The expectation:

$$\mu = E(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx = 0$$

$$\mu_2 = E(x^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2} dx$$: we use two-dimensional integral method:

$$(\mu_2)^2 = \int_{-\infty}^{\infty} \frac{1}{2\pi} x^2 e^{-x^2/2} dx \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy$$ we use polar-coordinate

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 y^2 e^{-(x^2+y^2)/2} dx\, dy$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} \rho^4 \sin^2\theta \cos^2\theta\, e^{-\frac{\rho^2}{2}} \cdot \rho\, d\rho\, d\theta$$

$$= \int_0^{\infty} \rho^5 e^{-\frac{\rho^2}{2}} d\rho \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{4} \frac{\cos 4\theta - 1}{2} d\theta$$

$$= \frac{1}{8} \int_0^{\infty} \rho^5 e^{-\frac{\rho^2}{2}} d\rho \quad (t = \frac{\rho^2}{2})$$

$$= \frac{1}{2} \int_0^{\infty} t^2 e^{-t} dt = \frac{1}{2} \Gamma(3) = 1$$

so $\mu^2 = 1$

$$\mu_4 = E(x^4) = \int_{-\infty}^{\infty} x^4 e^{-x^2/2}\, dx$$

$$\mu_4^2 = \frac{1}{2\pi} \iint \rho^9 \sin^4\theta \cos^4\theta\, e^{-\frac{\rho^2}{2}}\, d\rho\, d\theta$$

$$= \int_0^{\infty} \rho^9 e^{-\frac{\rho^2}{2}} d\rho \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{16} \left(\frac{\cos 2\theta - 1}{2}\right)^2 d\theta$$

$$= \int_0^{\infty} \rho^9 e^{-\frac{\rho^2}{2}} d\rho \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{16} \cdot \frac{1}{4} \left(\frac{1-\cos 8\theta}{2} - 2\cos 4\theta + 1\right) d\theta$$

$$= \frac{1}{16} \cdot \frac{3}{8} \cdot \int_0^{\infty} \rho^9 \cdot e^{-\frac{\rho^2}{2}} d\rho \quad \left(t = \frac{\rho^2}{2}\right)$$

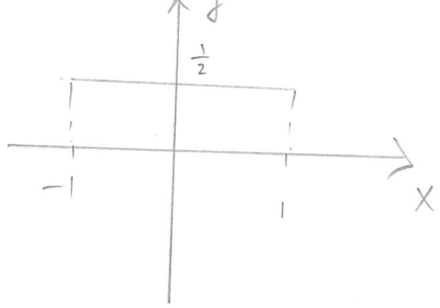
$$= \frac{3}{8} \int_0^{\infty} t^4 e^{-t} dt$$

$$= \frac{3}{8} \Gamma(5) = 9$$

$$\mu_4 = 3$$

$$\therefore \alpha_4 = \frac{\mu_4}{\mu_2^2} = 3$$

$$f(x) = \frac{1}{2} \qquad -1 < x < 1$$



The expectation:

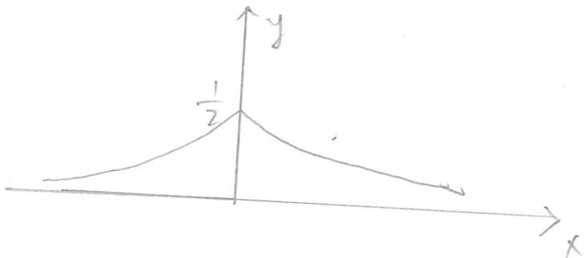
$$\mu = E(x) = \int_{-1}^{1} \frac{1}{2} x \, dx = 0$$

$$\mu_2 = E(x^2) = \int_{-1}^{1} \frac{1}{2} x^2 \, dx = \frac{1}{3}$$

$$\mu_4 = \int_{-1}^{1} \frac{1}{2} x^4 \, dx = \frac{1}{5}$$

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{9}{5}$$

$$f(x) = \frac{1}{2} e^{-|x|} \qquad -\infty < x < \infty$$



The expectation:

$$\mu = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} x \, dx = 0$$

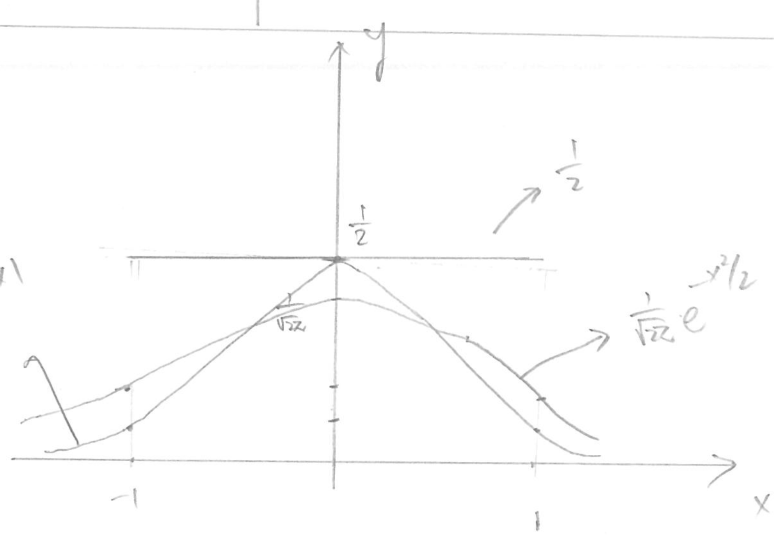
$$\mu_2 = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|} dx = \int_0^{\infty} x^2 e^{-x} dx = \Gamma(3) = 2! = 2$$

$$\mu_4 = \int_{-\infty}^{\infty} x^4 \frac{1}{2} e^{-|x|} dx = \int_0^{\infty} x^4 e^{-x} dx = \Gamma(5) = 4! = 24$$

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{24}{4} = 6$$

As we can see:

$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$\alpha_4 = 3$
$f_2(x) = \frac{1}{2}$	$\alpha_4 = \frac{9}{5}$
$f_3(x) = \frac{1}{2} e^{-\lvert x \rvert}$	$\alpha_4 = 6$



so α_4: $f_3 > f_1 > f_2$

and the peakedness of them is in the same order.