

Zhou Yuanting

321801314

Exercise 2-1.13

Let A be a symmetric operator.

There exist a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $D(A^*)$, such that $s\text{-}\lim_{n \rightarrow \infty} f_n = f$ and $s\text{-}\lim_{n \rightarrow \infty} A^* f_n = h$

Then For any $g \in D(A)$:

$$\langle f, Ag \rangle = \lim_{n \rightarrow \infty} \langle f_n, Ag \rangle = \lim_{n \rightarrow \infty} \langle A^* f_n, g \rangle = \langle h, g \rangle$$

hence $f \in D(A^*)$ and $A^* f = h$, which proves that A^* is closed.

proposition 1: If an operator A has a closed extension B , then A is closable and B is also an extension of the closure of A .

$$A \text{ is a symmetric operator} \Rightarrow \langle Af, g \rangle = \langle f, Ag \rangle$$

$$D(A) \text{ is a dense in } \mathcal{H} \text{ and } A \text{ is a symmetric} \Rightarrow \langle Af, g \rangle = \langle f, Ag \rangle$$

$$\therefore A \subset A^*$$

Indeed, there exist a vector f^* (namely $f^* = Af$) such that $\langle f, Ag \rangle = \langle f^*, g \rangle$.

since $A^* f = f^*$ by definition, one then has $A^* f = Af$.

$\therefore A^*$ is an extension of A

By proposition 1, A has a closed extension A^* , then A is closable.

\therefore A symmetric operator is always closable

Exercise 4.3.4

proof: There exist three vectors f_p, f_{sc}, f_{ac} in \mathcal{H} such that $f = f_p + f_{sc} + f_{ac}$

We shall see that the decomposition of the vector $f \in \mathcal{H}$ into $f = f_p + f_{sc} + f_{ac}$ induces an orthogonal decomposition of the Hilbert space \mathcal{H} into three subspaces that one denotes by $\mathcal{H}_p(A), \mathcal{H}_{sc}(A)$ and $\mathcal{H}_{ac}(A)$. The notation indicates that this decomposition depends on the considered spectral family $\{E_\lambda\}$, or equivalently on the self-adjoint operator A associated with this spectral family.

Thus each self-adjoint operator A in \mathcal{H} induces a decomposition of \mathcal{H} into

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A) \quad (1)$$

We denote the projections onto the subspaces appearing in $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A)$ by E_p, E_{sc}, E_{ac} .

Let $f \in \mathcal{D}(A)$ and write $f = f_p + f_{sc} + f_{ac} = E_p f + E_{sc} f + E_{ac} f$ in the first decomposition of \mathcal{H} in (1).

the measure m_f is the sum of three measures $m_{f_p}, m_{f_{sc}}$ and $m_{f_{ac}}$ determined by the three components of f : $m_f(V) = m_{f_p}(V) + m_{f_{sc}}(V) + m_{f_{ac}}(V) \quad \forall V \in \mathcal{A}_B$. (2)

by (2) one has

$$\int_{-\infty}^{\infty} \lambda^2 m_{f_p}(d\lambda) + \int_{-\infty}^{\infty} \lambda^2 m_{f_{sc}}(d\lambda) + \int_{-\infty}^{\infty} \lambda^2 m_{f_{ac}}(d\lambda) = \int_{-\infty}^{\infty} \lambda^2 m_f(d\lambda) < \infty$$

This implies that each of the vectors f_p, f_{sc} , and f_{ac} belong to $\mathcal{D}(A)$

We claim that $A f_p \in \mathcal{H}_p(A), A f_{sc} \in \mathcal{H}_{sc}(A)$ and $A f_{ac} \in \mathcal{H}_{ac}(A)$.

Let us verify for example the last inclusion: for each $M > 0$, let $\{\Pi_r^M\}$ be a sequence of partitions of the interval $(-M, M]$ with $|\Pi_r^M| \rightarrow 0$ as $r \rightarrow \infty$.

As defined as in $\Sigma(\Pi; \varphi) = \sum_{k=1}^N \varphi(u_k) E((s_{k-1}, s_k])$, $\Sigma(\Pi; \varphi)$ is a well defined operator

$\Pi = \Pi_r^M, \varphi(\lambda) = \lambda$, let Σ_r^M be the operator associated with the partition $\{\Pi_r^M\}$ and observe that this operator maps the subspace $\mathcal{H}_{ac}(A)$ into itself (because each $E(\Delta)$ has this property)

Thus $\Sigma_r^M f_{ac} \in \mathcal{H}_{ac}(A)$. Since $\mathcal{H}_{ac}(A)$ is a subspace, hence closed, one has

$$A f_{ac} = s\text{-}\lim_{M \rightarrow \infty} (s\text{-}\lim_{r \rightarrow \infty} \Sigma_r^M f_{ac}) \in \mathcal{H}_{ac}(A)$$

In a similar way one sees for example that $\varphi(A) f_{ac} \in \mathcal{H}_{ac}(A)$.

ii) The preceding considerations show that each $f \in D(A)$ has a unique decomposition $f = f_p + f_{sc} + f_{ac}$ with $f_p \in D(A_p)$, $f_{sc} \in D(A_{sc})$ and $f_{ac} \in D(A_{ac})$, and that $Af = A_p f_p + A_{sc} f_{sc} + A_{ac} f_{ac}$

This implies that, for $z \in \mathbb{K}$: $(A-z)f = (A_p-z)f_p + (A_{sc}-z)f_{sc} + (A_{ac}-z)f_{ac}$ (3)

iii) If $f, g \in D(A_p)$ then $\langle f, A_p g \rangle = \langle f, Ag \rangle = \langle Af, g \rangle = \langle A_p f, g \rangle$, then

the operators A_p is symmetric. Similarly, the operator A_{sc} and A_{ac} are symmetric.

To see that they are self-adjoint, it suffices to show that $R(A_p - i) = R(A_p + i) = \mathcal{H}_p(A)$

Now, since A is self-adjoint, we have $R(A \pm i) = \mathcal{H} \cap \text{ie}(A \pm i) D(A) = \mathcal{H}$

By (3) this is possible only if $(A_p \pm i)D(A_p) = \mathcal{H}_p(A)$, $(A_{sc} \pm i)D(A_{sc}) = \mathcal{H}_{sc}(A)$ and $(A_{ac} \pm i)D(A_{ac})$, which establishes the self-adjointness of A_p, A_{sc} and A_{ac} .

iv) Equation (3) shows that $A-z$ is invertible if and only if each of the three operators on its right-hand side is invertible, and if this is the case then one has $(A-z)^{-1} = (A_p-z)^{-1} \oplus (A_{sc}-z)^{-1} \oplus (A_{ac}-z)^{-1}$. This implies that

$z \in P(A)$ if and only if $z \in P(A)$ if and only if $z \in P(A_p) \cap P(A_{sc}) \cap P(A_{ac})$

It follows that $\sigma(A) = \mathbb{K} \setminus P(A) = \mathbb{K} \setminus \{P(A_p) \cap P(A_{sc}) \cap P(A_{ac})\}$

$$= [\mathbb{K} \setminus P(A_p)] \cup [\mathbb{K} \setminus P(A_{sc})] \cup [\mathbb{K} \setminus P(A_{ac})] = \sigma(A_p) \cup \sigma(A_{sc}) \cup \sigma(A_{ac})$$

Exercise 4.3.8

pf: Let us fix $\lambda \in \mathbb{R}$ and set $J_n = [\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]$ for $n=1, 2, 3, \dots$

a) Suppose that $\lambda \in \text{ess}(A)$, so that $\dim \mathcal{M}(J_n) = \infty$ for each n .

First take $f_1 \in \mathcal{M}(J_1)$ arbitrarily, with $\|f_1\| = 1$

Next choose a vector $f_2 \in \mathcal{M}(J_2)$ such that $f_2 \perp E(J_2)f_1$ and $\|f_2\| = 1$
then $\langle f_1, f_2 \rangle = \langle f_1, E(J_2)f_2 \rangle = \langle E(J_2)f_1, f_2 \rangle = 0$

Continue recursively in this way: $f_n \in \mathcal{M}(J_n)$ is chosen such that $\|f_n\| = 1$ and such that f_n is orthogonal to $E(J_n)f_1, \dots, E(J_n)f_{n-1}$.

One obtains in this way an orthonormal sequence $\{f_n\}$ (i.e. $\langle f_n, f_m \rangle = \delta_{nm}$) satisfying $E(J_n)f_n = f_n$

For any vector $g \in \mathcal{H}$: $\|g\|^2 \geq \sum_{n=1}^{\infty} |\langle f_n, g \rangle|^2$

consequently $\lim_{n \rightarrow \infty} \langle f_n, g \rangle = 0$.

$\therefore w - \lim_{n \rightarrow \infty} f_n = 0$

$$\|(A - \lambda)f_n\|^2 = \int_{J_n} |\lambda - \mu|^2 m_f(d\mu) \leq \frac{1}{n^2} \int_{-\infty}^{+\infty} m_f(d\mu) = \frac{1}{n^2} \|f_n\|^2 = \frac{1}{n^2}$$

Thus the sequence $\{f_n\}$ has the properties stated in the proposition

Conversely, suppose that there exist a sequence $\{f_n\}$ having these three properties, and assume that $\lambda \notin \text{ess}(A)$, i.e. that there exist a number $N \in \mathbb{N}$ s.t. $\dim \mathcal{M}(J_N) < \infty$

We must obtain a contradiction.

For this let us consider an interval (a, b) with $a < \lambda < b$.

$$\|(A - \lambda)f_n\|^2 = \int_{-\infty}^a (\lambda - \mu)^2 m_{f_n}(d\mu) + \int_b^{\infty} (\lambda - \mu)^2 m_{f_n}(d\mu) + \int_a^b |\lambda - \mu|^2 m_{f_n}(d\mu) \geq$$

$$(\lambda - a)^2 \|E((-\infty, a))f_n\|^2 + (b - \lambda)^2 \|E((b, \infty))f_n\|^2$$

Since $\|(A - \lambda)f_n\| \rightarrow 0$ as $n \rightarrow \infty$, this implies that $\|E((-\infty, a))f_n\| \rightarrow 0$ and $\|E((b, \infty))f_n\| \rightarrow 0$

$$\begin{cases} 1 = \|f_n\|^2 = \|E((-\infty, a))f_n\|^2 + \|E((b, \infty))f_n\|^2 + \|E([a, b])f_n\|^2 \\ \|E((-\infty, a))f_n\| \rightarrow 0 \text{ and } \|E((b, \infty))f_n\| \rightarrow 0 \end{cases} \Rightarrow \|E([a, b])f_n\| \rightarrow 1$$

In particular, taking $a = \lambda - \frac{1}{N}$, $b = \lambda + \frac{1}{N}$, we have $\|E(J_N)f_n\| \rightarrow 1$ as $n \rightarrow \infty$

If $\{e_1, \dots, e_M\}$ is an orthonormal basis of $\mathcal{M}(J_N)$, we have $\|E(J_N)f_n\|^2 = \sum_{k=1}^M |\langle e_k, f_n \rangle|^2$

Now $\langle e_k, f_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ for each fixed k because f_n converges weakly to zero, and the sum is finite by the assumption that $M = \dim \mathcal{M}(J_N) < \infty$

Hence $\|E(J_n)f_n\| \rightarrow 0$ as $n \rightarrow \infty$, in contradiction with the relation $\|E(J_n)f_n\| \rightarrow 1$

b) " \Rightarrow " In view of the result of (a), it suffices to consider the case $\lambda \in \overline{\sigma_p}(A)$

In this case one can take $f_n = f$ for each n , where f is a normalised eigenvector of A associated with the eigenvalue λ : $Af = \lambda f$, $\|f\| = 1$

" \Leftarrow " Let $a < \lambda < b$. The argument given in the second part of the proof of (a) shows that $\|E([a,b])f_n\| \rightarrow 1$ as $n \rightarrow \infty$, which implies that $\dim \mathcal{M}([a,b]) \neq 0$.

Hence λ is a point of non-constancy of the spectral family $\{E_\lambda\}$, i.e. $\lambda \in \sigma(A)$