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32190121 Kazumasa Narita

Exercise 4.1.1

Suppose that a function F satisfies the conditions (i) – (iii) on the page 43 of the lecture notes. We show that F has at most a countable set of points of discontinuity.

Define a family of sets $\{A_n\}_{n \in \mathbb{N}_{>0}}$ by

$$A_n := \{ \lambda \in \mathbb{R} \mid F(\lambda) - F(\lambda - 0) > \tfrac{1}{n} \}.$$

Each set A_n must be finite. In fact, set

$$\underline{\lambda}_n := \inf A_n, \quad \overline{\lambda}_n := \sup A_n.$$

(Note that $\underline{\lambda}_n$ and $\overline{\lambda}_n$ could be $-\infty$ and $+\infty$ respectively.)

Then, we have

$$F(\overline{\lambda}_n) > F(\underline{\lambda}_n) + \frac{1}{n} \times |A_n|,$$

where $|A_n|$ denotes the number of elements of A_n. By the assumption of F, $F(\overline{\lambda}_n)$ and $F(\underline{\lambda}_n)$ must be bounded and hence A_n must be finite.

The set of points of discontinuity can be written as $\bigcup_{n=1}^{\infty} A_n$. Hence, the claim follows.

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32190l211 Kazumasa Narita

This report is based on a part of Arai Asao's book 『ヒルベルト空間と量子力学』 ("Hilbert Space and Quantum Mechanics"), which is not translated or published in English.

Let F be a real-valued function on $\mathbb{R}$ which is bounded a.e. We denote the set of all Borel sets on $\mathbb{R}$ by $\mathcal{A}_B$. For each $B \in \mathcal{A}_B$, we define an operator $E_F(B)$ by

$$(E_F(B)f)(x) = \chi_{F^{-1}(B)}(x) f(x) \qquad x \in \mathbb{R},\ f \in L^2(\mathbb{R})$$

, where $F^{-1}(B)$ denotes the preimage of B by F, i.e.

$$F^{-1}(B) := \{x \in \mathbb{R} \mid F(x) \in B\}$$

① We show that E_F is a spectral measure.

By definition, it is clear that $E_F(B)$ is a bounded self-adjoint operator on $L^2(\mathbb{R})$. Furthermore, since we have $\chi_{F^{-1}(B)}(x)^2 = \chi_{F^{-1}(B)}(x)$, we obtain $E_F(B)^2 = E_F(B)$. Hence, for each $B \in \mathcal{A}_B$, $E_F(B)$ is an orthogonal projection on $L^2(\mathbb{R})$. In addition, we clearly have

$$E_F(\emptyset) = 0, \quad E_F(\mathbb{R}) = 1, \quad \text{and}$$
$$E_F(B_1)E_F(B_2) = E_F(B_1 \cap B_2).$$

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If $B \in \mathcal{A}_B$ and $\{B_j\}_{j=1}^{\infty} \subset \mathcal{A}_B$ satisfy the equations

$$B = \bigcup_{j=1}^{\infty} B_j, \quad B_i \cap B_j = \emptyset \quad (i \neq j),$$

then we have

$$F^{-1}(B) = \bigcup_{j=1}^{\infty} F^{-1}(B_j), \quad F^{-1}(B_i) \cap F^{-1}(B_j) = \emptyset \quad (i \neq j).$$

Therefore, for each $f \in L^2(\mathbb{R})$, we obtain

$$\| E_F(B) f - \sum_{j=1}^{N} E_F(B_j) f \|$$

$$= \int_{F^{-1}(B) \setminus \bigcup_{j=1}^{N} F^{-1}(B_j)} |f(x)|^2 \, dx$$

$$\longrightarrow 0 \quad (N \to \infty).$$

Hence, we obtain

$$E_F(B) = s\text{-}\lim_{N \to \infty} \sum_{j=1}^{N} E_F(B_j)$$

and conclude E_F is a spectral measure.

② Let $\mu_F(B) := m(F^{-1}(B))$, where m is the Lebesgue measure on $\mathbb{R}$. We show that μ_F is a measure on the measurable space $(\mathbb{R}, \mathcal{A}_B)$.

By definition, the range of μ_F is in $[0, \infty]$, and the equation $\mu_F(\emptyset) = 0$ holds.

PLUS

Furthermore, for B and B_j's satisfying

$$B = \bigcup_{j=1}^{\infty} B_j , \quad B_i \cap B_j = \emptyset \quad (i \neq j),$$

we have

$$\mu_F(B) = m(F^{-1}(B)) = m\left(\bigcup_{j=1}^{\infty} F^{-1}(B_j)\right)$$
$$= \sum_{j=1}^{\infty} m(F^{-1}(B_j))$$
$$= \sum_{j=1}^{\infty} \mu_F(B_j) .$$

Hence, μ_F is a measure on $(\mathbb{R}, \mathcal{A}_B)$.

③ Here, we add the assumption: F is a function such that $(\mathbb{R}, \mathcal{A}_B, \mu_F)$ is σ-finite. (For example, we can consider $F(x) = x$.)

We show that E_F is the spectral measure of the multiplication operator M_F.

For any $f, g \in L^2(\mathbb{R})$ and $B \in \mathcal{A}_B$, we have

$$\langle f, E_F(B) g \rangle = \int_{F^{-1}(B)} \overline{f(x)}\, g(x)\, dx.$$

Therefore, if $\mu_F(B) = m(F^{-1}(B)) = 0$, we have $\langle f, E_F(B) g \rangle = 0$. In other words, the complex measure $\langle f, E_F(\cdot) g \rangle$ is absolutely continuous with respect to μ_F.

Hence, Radon Nikodym theorem implies that there exists a function $\rho_{f,g}$ such that $|\rho_{f,g}|$ is measurable with respect to μ_F and the equation

$$\langle f, E_F(B) g \rangle = \int_B \rho_{f,g}(x)\, d\mu_F(x)$$

holds. Then,

$$\int_{F^{-1}(B)} \overline{f(x)}\, g(x)\, dx = \langle f, E_F(B) g \rangle$$

$$= \int_B \rho_{f,g}(y)\, d\mu_F(y)$$

$$= \int_{F^{-1}(B)} \rho_{f,g}(F(x))\, dx.$$

(☺ We changed the variable by $y = F(x)$)

Hence, it follows that

$$\overline{f(x)}\, g(x) = \rho_{f,g}(F(x)) \quad \text{a.e. } x.$$

Therefore, we obtain

$$\langle f, M_F g \rangle = \int_{\mathbb{R}} F(x)\, \overline{f(x)}\, g(x)\, dx$$

$$= \int_{\mathbb{R}} F(x)\, \rho_{f,g}(F(x))\, dx$$

$$= \int_{\mathbb{R}} \lambda\, \rho_{f,g}(\lambda)\, d\mu_F(\lambda).$$

□