

Exercise 1 For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ sufficiently differentiable, give the definition of the Hessian matrix and its main property.

Exercise 2 Consider the function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$ defined by $\phi(x, y) := (xy, \frac{y}{x})$. Compute its Jacobian matrix and its Jacobian determinant.

Exercise 3 For the following functions, compute their gradient:

$$f : \mathbb{R}^3 \ni (x, y, z) \mapsto e^{xyz} + 3y + 1 \in \mathbb{R}, \quad g : \mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto \sqrt{\sum_{j=1}^n x_j^2} \in \mathbb{R}.$$

Exercise 4 Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = xe^y$.

- (i) Compute its Taylor expansion around $(0, 0)$ of order 2 (a polynomial expression of degree 2) with an expression for the remainder term,
- (ii) Compute its Taylor expression around $(0, 0)$ of order 3, without providing any information on the remainder term.

Exercise 5 Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = 6xy^2 - 2x^3 - 3y^4$.

- (i) Determine the critical point(s) of f in \mathbb{R}^2 ,
- (ii) For each of the point(s) found in (i), determine if the point corresponds to a local maximum, a local minimum, or a saddle point. Justify your answer and try to go as far as possible in your analysis.

Exercise 6 Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^2 , let c be a real number, and set

$$F : \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{by} \quad F(x, t) = f(x - ct).$$

- (i) Show that F satisfies the equation $\partial_t^2 F - c^2 \partial_x^2 F = 0$, with $\partial_t^2 F = \partial_t[\partial_t F]$,
- (ii) Show that the function $\mathbb{R}^2 \ni (x, t) \mapsto f(x + ct) \in \mathbb{R}$ satisfies the same equation,
- (iii) Consider a second function $g : \mathbb{R} \rightarrow \mathbb{R}$ also of class C^2 , and define the function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$G(x, t) := \frac{1}{2c} \int_{x-ct}^{x+ct} g(r) dr + \frac{1}{2}(f(x + ct) + f(x - ct)).$$

Compute $G(x, 0)$, $[\partial_t G](x, 0)$ and $\partial_t^2 G - c^2 \partial_x^2 G$. What do you conclude ?

Exercise 1 3pt

$$H_f(X) = \begin{pmatrix} \partial_1^2 f(x) & \partial_2 \partial_1 f(x) & \dots & \partial_n \partial_1 f(x) \\ \partial_1 \partial_2 f(x) & \partial_2^2 f(x) & \dots & \partial_n \partial_2 f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 \partial_n f(x) & \partial_2 \partial_n f(x) & \dots & \partial_n^2 f(x) \end{pmatrix} \in M_{n \times n}(\mathbb{R})$$

This matrix is symmetric because $\partial_j \partial_k f(x) = \partial_k \partial_j f(x)$ if f is at least C^2 .

Exercise 2 3pt

$$D\phi(x, y) = \begin{pmatrix} \partial_x \phi_1(x, y) & \partial_y \phi_1(x, y) \\ \partial_x \phi_2(x, y) & \partial_y \phi_2(x, y) \end{pmatrix}$$

$$= \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix},$$

whose determinant is $\frac{y}{x} + x \frac{y}{x^2} = 2 \frac{y}{x}$. 1

Exercise 3 3pt

$$\nabla f(x, y, z) = \begin{pmatrix} yz e^{xyz} \\ xz e^{xyz} + 3 \\ xy e^{xyz} \end{pmatrix}. \quad 1\frac{1}{2}$$

$$\nabla g(x_1, \dots, x_n) = \frac{1}{\sqrt{\sum_{j=1}^n x_j^2}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad 1\frac{1}{2}$$

Exercise 4 4 pts

$$f(x, y) = x e^y$$

$\curvearrowright = (0,0)$

$$f(H) = f(0) + H \cdot \nabla f(0) + \frac{1}{2} H^T H f''(0) H + \text{Rem}$$

$$\text{One has : } \quad \partial_x f(0) = 1 \quad \partial_y f(0) = 0$$

$$\partial_x^2 f(0) = 0, \quad \partial_x \partial_y f(0) = 1 \quad \partial_y^2 f(0) = 0$$

$$\partial_x^3 f(H) = 0 \quad \partial_x^2 \partial_y f(H) = 0 \quad \partial_x \partial_y^2 f(H) = e^{h_2}$$

$$\partial_y^3 f(H) = h_1 e^{h_2}. \quad \text{Also } f(0) = 0$$

$$\text{Then } f(H) = h_1 + \frac{1}{2} 2 h_1 h_2 + \text{Rem}$$

$$= h_1 + h_1 h_2 + \text{Rem}$$

$$\text{with Rem} = \frac{1}{3!} [(H \cdot \nabla)^3 f](sH) \quad \text{for some } s \in (0,1).$$

$$\left(= \frac{1}{3!} (3 h_1 h_2^2 e^{sh_2} + sh_1 h_2^3 e^{sh_2}) \right).$$

For order 3 (without remainder)

$$f(H) \approx h_1 + h_1 h_2 + \frac{1}{3!} 3 h_1 h_2^2$$

$$= h_1 + h_1 h_2 + \frac{1}{2} h_1 h_2^2$$

Faster method :

$$f(x, y) = x e^y$$

$$= x (1 + y + \frac{1}{2} y^2 + O(y^3))$$

$$= x + xy + \frac{1}{2} x y^2 + \text{Rem.}$$

Exercise 5

8pt

$$f(x, y) = 6xy^2 - 2x^3 - 3y^4$$

$$\text{i) } \nabla f(x, y) = \begin{cases} 6y^2 - 6x^2 \\ 12xy - 12y^3 \end{cases}$$

$$\Rightarrow \nabla f(x, y) = (0) \iff \begin{cases} x^2 = y^2 \\ xy = y^3 \end{cases}$$

solution : $(0, 0)$, and if $y \neq 0$, $x = y^2 = x^2$
 $\Rightarrow x = 1$ and $y = \pm 1$.

3 solution : $(0, 0)$, $(1, 1)$, $(1, -1)$. 2

$$H_f(x, y) = \begin{pmatrix} -12x & 12y \\ 12y & 12x - 36y^2 \end{pmatrix}.$$

ii)

At $(0, 0)$: $H_f(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ \Rightarrow further analysis
 is necessary, see below. 1

$$\text{At } (1, 1) : H_f(1, 1) = \begin{pmatrix} -12 & 12 \\ 12 & -24 \end{pmatrix} = 12 \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\frac{1}{12} {}^T H H_f(1, 1) H = (h_1, h_2) \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (h_1, h_2) \begin{pmatrix} -h_1 + h_2 \\ h_1 - 2h_2 \end{pmatrix}$$

$$= -h_1^2 + 2h_1h_2 - 2h_2^2 = -(h_1^2 - 2h_1h_2 + h_2^2) - h_2^2$$

$$= -(h_1 - h_2)^2 - h_2^2 < 0 \quad \text{if } (h_1, h_2) \neq (0, 0)$$

$\Rightarrow (1, 1)$ is a local maximum. 2

$$\text{At } (1, -1) : H_f(1, -1) = \begin{pmatrix} -12 & -12 \\ -12 & -24 \end{pmatrix} = 12 \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

$$\frac{1}{12} {}^T H H_f(1, -1) H = (h_1, h_2) \begin{pmatrix} -h_1 - h_2 \\ -h_1 - 2h_2 \end{pmatrix} = -h_1^2 - 2h_1h_2 - 2h_2^2$$

$$= -(h_1 + h_2)^2 - h_2^2 < 0 \quad \text{if } (h_1, h_2) \neq (0, 0)$$

$\Rightarrow (1, -1)$ is a local maximum. 2

At $(0, 0)$: If we look at the diagonal $x=y$
we get

$$\begin{aligned} f(x, x) &= 6x^3 - 2x^3 - 3x^4 \\ &= 4x^3 - 3x^4 \\ &= x^3 \left(4 - 3x \right) \end{aligned}$$

\nearrow for $x \sim 0$

This changes its sign near $x=0$, which implies that $(0, 0)$ is a saddle point.

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Exercise 6 6 pts

/ composition

$$\begin{aligned} i) \quad \partial_t^2 F(x-ct) &= c^2 f''(x-ct) \\ \partial_x^2 F(x-ct) &= f''(x-ct) \quad / \text{ } 0 \text{ function} \\ \Rightarrow \quad \partial_t^2 F - c^2 \partial_x^2 F &= c^2 f'' - c^2 f'' = 0. \quad 2 \end{aligned}$$

ii) Same argument, since only c^2 is involved.

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$$iii) \quad G(x, 0) = \frac{1}{2c} \int_x^{2c} g(u) du + f(x) = \underline{f(x)}. \quad 1$$

$$\partial_t G(x, t) = \frac{1}{2c} \{ c g(x+ct) - (-c) g(x-ct) \}$$

$$+ \frac{1}{2} \{ c f'(x+ct) - c f'(x-ct) \}$$

$$\Rightarrow \partial_t G(x, 0) = \underline{g(x)}. \quad 1$$

$$\partial_t^2 G(x, t) = \frac{1}{2} \left\{ c g'(x+ct) - c g'(x-ct) \right\}$$

$$+ \frac{1}{2} c^2 \left\{ f''(x+ct) + f''(x-ct) \right\} .$$

$$\partial_x^2 G(x, t) = \frac{1}{2c} \left\{ g'(x+ct) - g'(x-ct) \right\}$$

$$+ \frac{1}{2} \left\{ f''(x+ct) + f''(x-ct) \right\}$$

$$\Rightarrow \partial_t^2 G - c^2 \partial_x^2 G = 0.$$

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Thus G is a solution of $\partial_t^2 G - c^2 \partial_x^2 G = 0$

with initial condition $G(0, 0) = f$
 $\partial_t G(0, 0) = g$.