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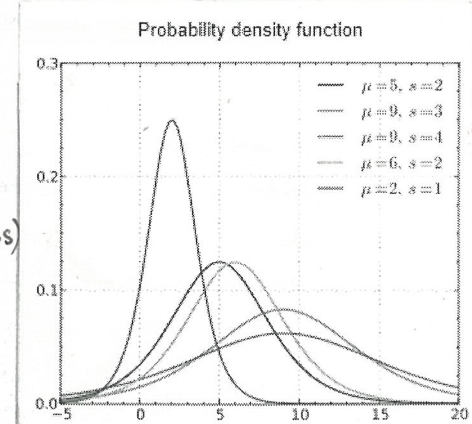
Logistic distribution

Logistic distribution is a continuous probability distribution.

Parameters	Location parameter: μ, $-\infty<\mu<\infty$ Scale parameter: $s>0$	Mean	$E(X)=\mu$
Support	$-\infty<x<\infty$	Mode	μ
Pdf	$f(x\mid\mu;s)=\dfrac{\exp[-(x-\mu)/s]}{s\{1+\exp[-(x-\mu)/s]\}^2}=\dfrac{1}{4s}\operatorname{sech}^2\left(\dfrac{x-\mu}{2s}\right)$	Median	μ
		Variance	$Var(X)=\dfrac{s^2\pi^2}{3}$
Cdf	$F(x\mid\mu;s)=\dfrac{1}{1+\exp[-(x-\mu)/s]}=\dfrac{1}{2}+\dfrac{1}{2}\tanh\left(\dfrac{x-\mu}{2s}\right)$	Coefficient of skewness	0
Mgf	$M_X(t)=e^{\mu t}\,\Gamma(1-st)\,\Gamma(1+st)=\dfrac{\pi st\,e^{\mu t}}{\sin(\pi st)}$ $\left(\lvert t\rvert<\frac{1}{s}\right)$	Coefficient of kurtosis	4.2
CF (characteristic function)	$\varphi_X(t)=e^{it\mu}\,\dfrac{\pi st}{\sinh(\pi st)}$		
Quantile function	$Q(p\mid\mu;s)=\mu+s\ln\left(\dfrac{p}{1-p}\right)$		
Quantile density function	$Q'(p\mid s)=\dfrac{s}{p(1-p)}$		



⊛ Related distributions:

+ Logistic distribution mimics the sech distribution
+ If $X\sim \text{Logistic}(\mu;s)$ then $kX+\ell\sim \text{Logistic}(k\mu+\ell;\ ks)$
+ If $X\sim U(0;1)$ then $\mu+s[\log(X)-\log(1-X)]\sim \text{Logistic}(\mu;s)$
+ If $X\sim \text{Gumbel}(\alpha_X;s)$ and $Y\sim \text{Gumbel}(\alpha_Y;s)$ then $X-Y\sim \text{Logistic}(\alpha_X-\alpha_Y;s)$
+ If $X\sim \text{Logistic}(\mu;s)$ then $\exp(X)\sim \text{LogLogistic}\left(\alpha=e^{\mu},\ \beta=\frac{1}{s}\right)$ and $\exp(X)+\gamma\sim$ shifted log-logistic $\left(\alpha=e^{\mu};\beta=\frac{1}{s};\gamma\right)$
+ If $X\sim \text{Exponential}(1)$ then $\mu+s\log(e^{X}-1)\sim \text{Logistic}(\mu;s)$
+ If $X;Y\sim \text{Exponential}(1)$ then $\mu-s\log\left(\frac{X}{Y}\right)\sim \text{Logistic}(\mu;s)$

⊙ The pdf curves are symmetric about $x=\mu$ and unimodal (having a single peak).
⊙ If we fix μ and increase s, then the height decreases and the curve "spreads" more widely. If we fix s and increase μ, then the curve only shifts to the right without any changes in height or width.
⊙ The Logistic distribution looks similar to the Normal distribution but has heavier tails.

⊛ n^{th}-order central moment:

$$E[(X-\mu)^n]=\int_{-\infty}^{\infty}(x-\mu)^n f(x)\,dx=\int_0^1[Q(p)-\mu]^n dp=s^n\int_0^1\left[\ln\left(\frac{p}{1-p}\right)\right]^n dp$$

⊛ An example of logistic distribution:

Logistic pdf is applied in aphid population growth model in ecology. The model describes a common type of growth curve in which a population rises to some maximum value and then decreases quickly.

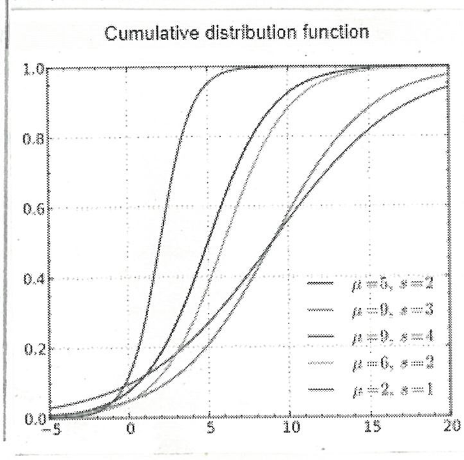
Let $N(t)$ denote population size at time t

According to Kindlmann (1985), insect populations are described by:

$$N'(t)=(\lambda-\delta F(t)).N(t) \quad \text{with past population size } F(t)=\int_0^t N(s)\,ds$$

The solution is: $$N(t)=\frac{4N_{max}e^{-b(t-t_{max})}}{[1+e^{-b(t-t_{max})}]^2}$$ $\begin{cases} N_{max}: \text{maximum size of } N(t) \\ t_{max}: \text{time of } N_{max}\end{cases}$

[Matis et al. (2007)]



S-shaped

⊛ Other applications:

Logistic regression; Fermi-Dirac statistics in quantum mechanics; Hydrology; Chess ratings; etc.

Sources:

(1) Wikipedia

(2) "Theory and Application of the Logistic Probability Density Function as a Population Growth Model", J.H. Matis and M.J. Al-Muhammed, 2010

The cdf is also called the logistic function:

$$F(x|\mu;s) = \frac{1}{1+\exp(-\frac{x-\mu}{s})} = \frac{\exp(\frac{x-\mu}{s})}{\exp(\frac{x-\mu}{s})+1}$$

Origin: first introduced by P.F. Verhulst in 1838~1847 as a model of population growth

Let N be the population size, t be the time, r be the growth rate and K be the carrying capacity, one has the differential equation of population growth:

$$N'(t) = rN\left(1-\frac{N}{K}\right)$$

The solution is the logistic function times K:

$$N(t) = \frac{K}{1+(\frac{K}{N(0)}-1)\exp(-rt)} = \frac{K}{1+\exp[-rt+\log(\frac{K}{N(0)}-1)]} = KF\left(t\,\Big|\,\frac{1}{r}\log\left(\frac{K}{N(0)}-1\right);\frac{1}{r}\right)$$

Some properties of logistic function

- $F(x|\mu;s) \xrightarrow{x\to-\infty} \exp\left(\frac{x-\mu}{s}\right)$
- $F(x|\mu;s) \xrightarrow{x\to\infty} 1-\exp\left(-\frac{x-\mu}{s}\right)$
- $F(x|\mu;s)\big|_{x=\mu} = \frac{1}{2}$; $\frac{\partial F}{\partial x}(x|\mu;s)\big|_{x=\mu} = \frac{1}{2s}$; but $\frac{\partial^2 F}{\partial x^2}(x|\mu;s)\big|_{x=\mu} = 0$

$\Rightarrow$ F behaves almost linearly around $x=\mu$.

Similar distributions

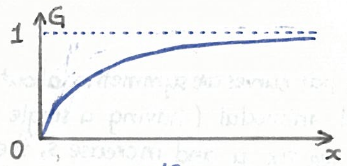
⊙ Half-logistic distribution:

the distribution of $X := |Y|$ where $Y \sim \text{Logistic}(0;s)$ with cdf $F(y)$ and pdf $f(y)$

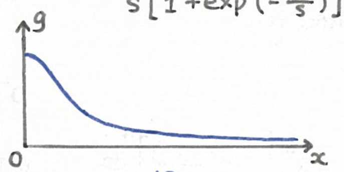
support: $[0,+\infty)$

cdf: $G(x) = P(\{X\le x\}) = P(\{-x\le Y\le x\}) = F(x)-F(-x) = 2F(x)-1 = \frac{1-\exp(-\frac{x}{s})}{1+\exp(-\frac{x}{s})}$

pdf: $g(x) = f(x)+f(-x) = 2f(x) = \frac{2\exp(-\frac{x}{s})}{s[1+\exp(-\frac{x}{s})]^2}$



cdf



pdf

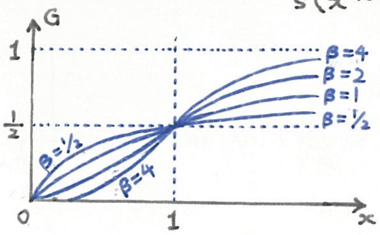
⊙ Log-logistic distribution:

the distribution of X with $\log(X) =: Y \sim \text{Logistic}(\mu;s)$ with cdf $F(y)$ and pdf $f(y)$

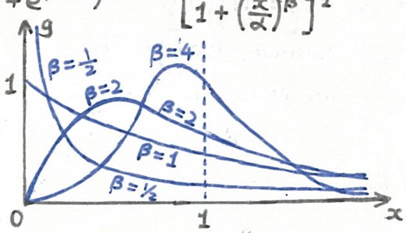
support: $[0,+\infty)$

cdf: $G(x) = P(\{X\le x\}) = P(\{Y\le\log(x)\}) = F(\log(x)) = \frac{x^{1/s}}{x^{1/s}+e^{\mu/s}} = \frac{1}{1+(\frac{x}{\alpha})^{-\beta}}$ $\quad\begin{pmatrix}\alpha = e^{\mu}\\ \beta=\frac{1}{s}\end{pmatrix}$

pdf: $g(x) = G'(x) = \frac{x^{1/s-1}e^{\mu/s}}{s(x^{1/s}+e^{\mu/s})^2} = \frac{(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}}{[1+(\frac{x}{\alpha})^{\beta}]^2}$



cdf for $\alpha=1$, $\mu=0$



pdf for $\alpha=1$, $\mu=0$