Zhang Liyang   Statistics Exercise

Lecture 3, p. 7

Lemma: $|\rho_{XY}| = 1 \Leftrightarrow \exists a, b \in \mathbb{R}: P(Y = aX + b) = 1$

Proof: For easy distinguishment we set

$\text{积}(\equiv\text{積}) := E(XY)$; $\text{甲} := E(X^2)$; $\text{乙} := E(Y^2)$;

$\text{天} := E(X)$; $\text{地} := E(Y)$

(1) 「⇒」 From $|\rho_{XY}| = 1 \Leftrightarrow \frac{Cov(X,Y)}{\sigma_X \sigma_Y} =: \rho_{XY} = \pm 1 \Leftrightarrow Cov(X,Y)^2 = \sigma_X^2 \sigma_Y^2$

Since $Cov(X,Y)^2 = [E(XY) - E(X)E(Y)]^2 = (\text{积} - \text{天地})^2$

and $\sigma_X^2 \sigma_Y^2 = [E(X^2) - E(X)^2][E(Y^2) - E(Y)^2] = (\text{甲} - \text{天}^2)(\text{乙} - \text{地}^2)$

Then $(\text{积} - \text{天地})^2 = (\text{甲} - \text{天}^2)(\text{乙} - \text{地}^2)$ (*)

Since $\sigma_{aX+b}^2 = a^2 \sigma_X^2 = a^2(\text{甲} - \text{天}^2)$, and $Cov(X, aX+b)$ has same sign with a, set

$a := sgn(Cov(X,Y)) \sqrt{\sigma_Y^2 / \sigma_X^2} = sgn(\text{积} - \text{天地}) \sqrt{\sigma_Y^2 - \sigma_X^2}$

$\Rightarrow a^2 = \frac{\sigma_Y^2}{\sigma_X^2} = \frac{\text{乙} - \text{地}^2}{\text{甲} - \text{天}^2} \Rightarrow \text{乙} - \text{地}^2 = a^2(\text{甲} - \text{天}^2)$

From (*),

$(\text{积} - \text{天地})^2 = (\text{甲} - \text{天}^2)(\text{乙} - \text{地}^2) = a^2(\text{甲} - \text{天}^2)^2$

$\Rightarrow \text{积} - \text{天地} = \pm a (\text{甲} - \text{天}^2)$

$\Rightarrow (\text{积} - \text{天地}) a = \pm a^2 (\text{甲} - \text{天}^2) = \pm a^2 \sigma_X^2$ (+)

Since $a(\text{积} - \text{天地}) = sgn(\text{积} - \text{天地})^2 |\text{积} - \text{天地}| \sqrt{\sigma_Y^2 - \sigma_X^2} > 0$ and $a^2 \sigma_X^2 > 0$,

The sign in equation (+) should be positive.

$\Rightarrow (\text{积} - \text{天地}) a = + a^2 (\text{甲} - \text{天}^2)$

$$
\begin{aligned}
\Rightarrow \sigma_{Y-aX}^2 &= E[(Y - aX)^2] - E(Y - aX)^2 \\
&= E(Y^2 - 2aXY + a^2X^2) - (E(Y) - aE(X))^2 \\
&= E(Y^2) - 2aE(XY) + a^2E(X^2) - E(Y)^2 + 2aE(X)E(Y) - a^2E(X)^2 \quad \text{linearity easy to prov} \\
&= \text{乙} - 2a\,\text{积} + a^2\,\text{甲} - \text{地}^2 + 2a\,\text{天}\,\text{地} - a^2\,\text{天}^2 \\
&= a^2(\text{甲} - \text{天}^2) - 2a(\text{积} - \text{天地}) + (\text{乙} - \text{地}^2) \\
&= a^2(\text{甲} - \text{天}^2) - 2a^2(\text{甲} - \text{天}^2) + a^2(\text{甲} - \text{天}^2) \\
&= 0
\end{aligned}
$$

Set $b := E(Y - aX)$

Since $\sigma_{Y-aX}^2 := E[(Y - aX - b)^2]$ with appearantly $(Y - aX - b)^2 \geq 0$,

$\sigma_{Y-aX}^2 = 0$ shows that $P[(Y - aX - b)^2 = 0] = 1$ Otherwise $\sigma_{Y-aX}^2 = E[(Y-aX-b)]^2 > 0$

$\Rightarrow P(Y - aX - b = 0) = 1$

$\Rightarrow P(Y = aX + b) = 1$

(2) 「$\Leftarrow$」 From $Y = aX + b$

$$\begin{aligned}
\Rightarrow \mathrm{Cov}(X,Y) &= \mathrm{Cov}(X, aX+b) \\
&= E[X(aX+b)] - E(X)E(aX+b) \\
&= E(aX^2 + bX) - E(X)[aE(X)+b] \\
&= aE(X^2) + bE(X) - aE(X)^2 - bE(X) \quad \text{linearity easy to prove} \\
&= a[E(X^2) - E(X)^2] \\
&= a\sigma_X^2
\end{aligned}$$

Since $\sigma_Y = \sigma_{aX+b} = |a|\sigma_X$,

$$\mathrm{Cov}(X,Y) = \mathrm{sgn}(a)\,\sigma_X \sigma_Y$$

$$\Rightarrow \rho_{XY} := \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \mathrm{sgn}(a)$$

$$\Rightarrow |\rho_{XY}| = 1$$