

Exercise 4.1.1. Show that any monotonically non-decreasing function $F: \mathbb{R} \rightarrow \mathbb{R}$ has at most a countable set of points of discontinuity.

Proof. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be monotonically non-decreasing. Then for any $x \in \mathbb{R}$, $\{F(y) | y < x\}$ is apparently non-empty and upper bounded by $F(x)$. Therefore, $\sup_{y < x} F(y)$ is a real number, instead of $\pm\infty$. Hence, by the definition of supremum, $\forall \varepsilon > 0 \exists y' < x$ s.t. $F(y') > \sup_{y < x} F(y) - \varepsilon$. Thus, due to the monotonicity of F , $\exists z \in (y', x)$,

$$|F(z) - \sup_{y < x} F(y)| = \sup_{y < x} F(y) - F(z) \leq \sup_{y < x} F(y) - F(y') < \varepsilon.$$

Hence, we have so far shown that $\forall \varepsilon > 0 \exists \delta > 0 \forall z \in (x-\delta, x)$, $|\sup_{y < x} F(y) - F(z)| < \varepsilon$.

By the definition of left limit, $\sup_{y < x} F(y) = \lim_{y \rightarrow x^-} F(y)$.

By the same argument, $\inf_{y > x} F(y)$ is also finite and equals $\lim_{y \rightarrow x^+} F(y)$.

As F is monotonically non-decreasing, $\lim_{y \rightarrow x} F(y) = \sup_{y < x} F(y) \leq \inf_{y > x} F(y) = \lim_{y \rightarrow x^+} F(y)$ holds for any $x \in \mathbb{R}$, with equality if and only if F is continuous at x . Hence, to each point x of discontinuity we can associate an open interval $(\lim_{y \rightarrow x} F(y), \lim_{y \rightarrow x^+} F(y))$, which is non-empty.

Additionally, by the monotonicity of F , whenever $x < x'$, $\lim_{y \rightarrow x^+} F(y) \leq \lim_{y \rightarrow x'^+} F(y)$.

Therefore, any two different points of discontinuity are associated with two disjoint open intervals. As \mathbb{Q} is dense in \mathbb{R} , $\exists q_x \in \mathbb{Q} \cap (\lim_{y \rightarrow x} F(y), \lim_{y \rightarrow x^+} F(y))$. Since these open intervals are pairwise disjoint, $x \mapsto q_x$ is injective. Therefore,

$$|\{\text{points of discontinuity of } F\}| \leq |\mathbb{Q}|.$$

As \mathbb{Q} is countable, any monotonically non-decreasing function $F: \mathbb{R} \rightarrow \mathbb{R}$ has at most countably many points of discontinuity. ◻

Two Exercises in Chap. 4.

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Exercise 4.2.9. Prove that, for any self-adjoint operator A , $\text{supp}\{E_\lambda\} = \sigma(A)$, where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ denotes the spectral family associated to A .

Proof $\forall z \in p(A)$, consider $\int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda)$ with some appropriate domain D .

$$\text{Note that } (A - z) \int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda) = (\int_{-\infty}^{+\infty} \lambda E(d\lambda) - \int_{-\infty}^{+\infty} z E(d\lambda)) \int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda) = \int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda) f^{(z)}(\lambda).$$

$$\int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda) f^{(z)}(\lambda) = \int_{-\infty}^{+\infty} E(d\lambda)|_D = \text{id}_D. \text{ Similarly, } \int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda) (A - z) = \lambda \text{id}_D(A).$$

Therefore, as $z \in p(A)$, $\int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda)$ can be extended to the bounded operator $(A - z)^{-1}$.

Hence, $\|\int_{-\infty}^{+\infty} (\lambda - z)^{-1} E(d\lambda)\| \leq M$ for some $M > 0$. By the property of spectral family which states that $\|\int_a^b (\lambda - z)^{-1} E(d\lambda)\| = \sup_{\mu \in [a, b] \cap \text{supp}\{E_\lambda\}} |\lambda - z|^{-1}$, we obtain that, if $z \in \mathbb{R}$,

$(z - \frac{1}{M}, z + \frac{1}{M})$ is a subset of the complement of $\text{supp}\{E_\lambda\}$, implying that $z \notin \text{supp}\{E_\lambda\}$. If $z \notin \mathbb{R}$, as $\text{supp}\{E_\lambda\}$ is by definition a subset of \mathbb{R} , $z \notin \text{supp}\{E_\lambda\}$. Hence, we conclude that $\text{supp}\{E_\lambda\} \subseteq \sigma(A)$.

$\forall z \in \sigma(A)$, z must be real for A is self-adjoint and there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ with $\|f_n\| = 1$ for any $n \in \mathbb{N}$ such that $\|(A - z)f_n\| \rightarrow 0$ as $n \rightarrow +\infty$. $\|(A - z)f_n\|$ can be calculated as follows: $\|\int_{-\infty}^{+\infty} (\lambda - z) E(d\lambda) f_n\|^2 = \lim_{M \rightarrow +\infty} \int_{-M}^M (\lambda - z)^2 E(d\lambda) f_n^2 \stackrel{(a)}{\leq} \lim_{M \rightarrow +\infty} \|\int_{-M}^M (\lambda - z)^2 E(d\lambda) f_n\|^2 =$

$$\lim_{M \rightarrow +\infty} \int_{-M}^M |\lambda - z|^2 m_{f_n}(d\lambda) \stackrel{(b)}{=} \int_{-\infty}^{+\infty} |\lambda - z|^2 m_{f_n}(d\lambda), \text{ where } m_{f_n}(d\lambda) = \langle f_n, E(d\lambda) f_n \rangle,$$

(a) follows from the definition of strong limit, and (b) follows from monotone convergence theorem. Note that

$$z \in \mathbb{R}. \text{ Then, } \int_{-\infty}^{z-\varepsilon} \lambda^{-2} m_{f_n}(d\lambda) + \int_{z+\varepsilon}^{+\infty} \lambda^{-2} m_{f_n}(d\lambda) = \frac{1}{\varepsilon^2} \left(\int_{-\infty}^{z-\varepsilon} \varepsilon^2 E(d\lambda) + \int_{z+\varepsilon}^{+\infty} \varepsilon^2 E(d\lambda) \right)$$

$$\leq \frac{1}{\varepsilon^2} \left(\int_{-\infty}^{z-\varepsilon} |\lambda - z|^2 m_{f_n}(d\lambda) + \int_{z+\varepsilon}^{+\infty} |\lambda - z|^2 m_{f_n}(d\lambda) \right) \leq \frac{1}{\varepsilon^2} \int_{-\infty}^{+\infty} |\lambda - z|^2 m_{f_n}(d\lambda) \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

$$\text{whereas } \forall n \in \mathbb{N}, \int_{-\infty}^{+\infty} m_{f_n}(d\lambda) = \|\int_{-\infty}^{+\infty} E(d\lambda) f_n\|^2 = \|f_n\|^2 = 1. \text{ Hence, } \forall \varepsilon > 0, \int_{z-\varepsilon}^{z+\varepsilon} m_{f_n}(d\lambda) \rightarrow 1$$

as $n \rightarrow +\infty$. Hence, $z \notin \text{supp}\{E_\lambda\}$. Consequently, $\sigma(A) \subseteq \text{supp}\{E_\lambda\}$.

In brief, $\sigma(A) = \text{supp}\{E_\lambda\}$.

