Exercises in Chap. 3

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Exercise 3.1.2. For any real $\varphi \in C(\mathbb{R}^d)$, show that the spectrum of the self-adjoint multiplication operator $\varphi(X)$ coincides with the closure of $\varphi(\mathbb{R}^d)$ in $\mathbb{R}$.

In addition, for any real $\varphi \in L^\infty(\mathbb{R}^d)$, show that the spectrum of the self-adjoint multiplication operator $\varphi(X)$ coincides with the essential closure of $\varphi(\mathbb{R}^d)$ in $\mathbb{R}$.

Proof. As is apparent by the definition, whenever $\varphi(X) - \lambda$ is invertible, the inverse is defined as $f \mapsto (\varphi(X) - \lambda)^{-1} f$, where f is a L^2 function, for which the function $x \mapsto (\varphi(x) - \lambda)^{-1} f(x)$ is still in $L^2(\mathbb{R}^d)$.

When $\lambda \in \mathbb{R}$ satisfies that there exists a positive ρ for which $|\varphi(x) - \lambda| \geq \rho$ a.e., $(\varphi(x) - \lambda)^{-1}$ is well-defined a.e. and

$$\int_{\mathbb{R}^d} \left| \frac{f(x)}{\varphi(x) - \lambda} \right|^2 \mathrm{d}x \leq \frac{1}{\rho^2} \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x.$$

Note that this inequality implies that

1. for any $f \in L^2(\mathbb{R}^d)$, $(\varphi(X) - \lambda)^{-1} f \in L^2(\mathbb{R}^d)$, which is to say that the inverse of $\varphi(X) - \lambda$ is defined on the entire space;
2. the norm of $(\varphi(X) - \lambda)^{-1}$ is well-defined and not greater than ρ^{-1}.

Combining those results, one obtains that λ is in the resolvent set of the multiplication operator in this case.

Conversely, assume that for a fixed $\lambda \in \mathbb{R}$, for any $\rho > 0$, $E_\rho := \{x \mid |\varphi(x) - \lambda| < \rho\}$ is of positive measure, where by convention the infinity is also regarded positive. Under such circumstances, we claim that λ is in the spectrum of the multiplication operator. In order to show this, it suffices to consider the case where $\varphi(x) \neq \lambda$ a.e.. Take a subset A of E_ρ satisfying that the measure of A is both positive and finite. Let f be the indicator of A, i.e.

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

thus making f in $L^2(\mathbb{R}^d)$ and different from zero. Note that

$$\int_{\mathbb{R}^d} \left| \frac{f(x)}{\varphi(x) - \lambda} \right|^2 \mathrm{d}x = \int_A \left| \frac{f(x)}{\varphi(x) - \lambda} \right|^2 \mathrm{d}x \geq \rho^{-2} \int_A |f(x)|^2 \mathrm{d}x = \rho^{-2} \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x.$$

Since ρ is an arbitrary positive number, $(\varphi(X) - \lambda)^{-1}$ is not bounded even if it exists. Therefore, in this case, λ is in the spectrum of the multiplication operator. $\square$

Exercise 3.1.4. Show that the following relations hold on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$:

$$[iX_j, X_k] = 0 = [D_j, D_k]$$

for any $j,\ k \in \{1, \cdots, d\}$ while

$$[iD_j, X_k] = \delta_{jk}.$$

Proof. By the definition of the Schwartz space, the operators involved are all defined on the entire Schwartz space. Thus, it suffices to check the equations by performing those operators on an arbitrary function in the Schwartz space.

Let $f \in \mathcal{S}(\mathbb{R}^d)$. By definition, one obtains for any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$([iX_j, X_k]f)(x) = (iX_jX_kf)(x) - (iX_kX_jf)(x) = ix_jx_kf(x) - ix_kx_jf(x) = 0.$$

As a result, $[X_j, X_k]f = 0$ holds, and by applying Fourier transform and its inverse, one has

$$\begin{aligned}
[D_j, D_k]f &= D_jD_kf - D_kD_jf \\
&= \mathcal{F}^*X_j\mathcal{F}\mathcal{F}^*X_k\mathcal{F}f - \mathcal{F}^*X_k\mathcal{F}\mathcal{F}^*X_j\mathcal{F}f \\
&= \mathcal{F}^*[X_j, X_k]\mathcal{F}f \\
&= 0.
\end{aligned}$$

By exploiting the fact that $D_j = -i\partial_j$, it is straightforward that for any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$([iD_j, X_k]f)(x) = \partial_jx_kf(x) - x_k\partial_jf(x) = 0,$$

whenever $j \neq k$. Moreover, for any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$([iD_j, X_j]f)(x) = \partial_jx_jf(x) - x_j\partial_jf(x) = f(x) + x_j\partial_jf(x) - x_j\partial_jf(x) = f(x),$$

which illustrates that $[iD_j, X_j] = 1$. In brief, one can conclude that $[iD_j, X_k] = \delta_{jk}$. □

Exercise 3.3.1. Check that if $f(x, \xi) = f(\xi)$, i.e. f is independent of x, then $\mathfrak{Op}(f) = f(D)$, while if $f(x, \xi) = f(x)$, i.e. f is independent of ξ, then $\mathfrak{Op}(f) = f(X)$.

Proof. We shall assume that $x \mapsto f(x)$, $\xi \mapsto f(\xi)$ and u are all in the Schwartz space so that the following calculations are valid.

When $f(x, \xi) = f(\xi)$,

$$\begin{aligned}
[\mathfrak{Op}(f)u](x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} e^{i(x-y)\cdot\eta} f(\eta)\, u(y) \mathrm{d}\eta \mathrm{d}y \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(y) \frac{1}{(2\pi)^{d/2}} \int_{\widehat{\mathbb{R}^d}} e^{i(x-y)\cdot\eta} f(\eta)\, \mathrm{d}\eta \mathrm{d}y \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(y) \check{f}(x - y) \mathrm{d}y \\
&= \check{f}(x) * u(x) \\
&= [f(D)u](x),
\end{aligned}$$

implying that $\mathfrak{Op}(f) = f(D)$ when f is independent of x.

When $f(x,\xi) = f(x)$,

$$\begin{aligned}[\mathfrak{Op}(f)u](x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}\right)u(y)\mathrm{d}\eta\mathrm{d}y \\ &= f\left(\frac{x+x}{2}\right)u(x) = f(x)u(x),\end{aligned}$$

where the second equality is obtained by first performing Fourier transform with respect to y and then performing the inverse Fourier transform with respect to η. $\square$

Exercise 3.3.2. The following holds for any Schwartz function f and g:

1. Let $\mathcal{F}_\Xi$ denote the symplectic Fourier transform. Then,

$$\mathfrak{Op}(f) = \frac{1}{(2\pi)^d}\int_\Xi (\mathcal{F}_\Xi^{-1}f)(x,\xi)W(x,\xi)\mathrm{d}\xi\mathrm{d}x.$$

2. $\mathfrak{Op}(f)\mathfrak{Op}(g) = \mathfrak{Op}(f\circ g)$.

Proof. 1. First we show by direct calculation that $\mathcal{F}_\Xi^2$ is the identity map, meaning that $\mathcal{F}_\Xi^{-1} = \mathcal{F}_\Xi$.

$$\begin{aligned}\mathcal{F}_\Xi^2(f)(x,\xi) &= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(y\cdot\xi - x\cdot\eta + z\cdot\eta - y\cdot\zeta)} f(z,\zeta)\mathrm{d}\zeta\mathrm{d}z\mathrm{d}\eta\mathrm{d}y \\ &\overset{(a)}{=} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(z-x)\cdot\eta} f(z,\xi)\mathrm{d}\eta\mathrm{d}z \\ &\overset{(b)}{=} f(x,\xi),\end{aligned}$$

where we perform the Fourier transform and the inverse Fourier transform with respect to ζ and y respectively in (a), and similarly, the inverse Fourier transform and the Fourier transform are successively performed with respect to z and η in (b).

Therefore, it follows that

$$\begin{aligned}&\left(\left(\frac{1}{(2\pi)^d}\int_\Xi (\mathcal{F}_\Xi^{-1}f)(x,\xi)W(x,\xi)\mathrm{d}\xi\mathrm{d}x\right)u\right)(z) \\ &= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(y\cdot\xi - x\cdot\eta)} f(y,\eta)e^{\frac{i}{2}x\cdot\xi}U_xV_\xi u(z)\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi\mathrm{d}x \\ &= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(y\cdot\xi - x\cdot\eta)} f(y,\eta)e^{\frac{i}{2}x\cdot\xi}e^{-i(z+x)\cdot\xi}u(z+x)\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi\mathrm{d}x \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{-ix\cdot\eta} f\left(z+\frac{x}{2},\eta\right)u(z+x)\mathrm{d}\eta\mathrm{d}x \\ &\overset{(c)}{=} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{-i(x-z)\cdot\eta} f\left(z+\frac{x-z}{2},\eta\right)u(x)\mathrm{d}\eta\mathrm{d}x \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} e^{i(z-x)\cdot\eta} f\left(\frac{x+z}{2},\eta\right)u(x)\mathrm{d}\eta\mathrm{d}x = (\mathfrak{Op}(f)u)(z),\end{aligned}$$

where (c) is obtained by changing of coordinates, or alternatively due to the invariance of Lebesgue measure under translation.

2. We shall first calculate the following for $\mathrm{x} = (x, \xi)$, $\mathrm{y} = (y, \eta)$, z and w in $\Xi = \mathbb{R}^d \times \widehat{\mathbb{R}^d}$:

$$
\begin{aligned}
\mathcal{F}_\Xi(f \circ g)(\mathrm{x}) &= \frac{4^d}{(2\pi)^{3d}} \int_\Xi \int_\Xi \int_\Xi e^{i\sigma(\mathrm{x},\mathrm{w})} e^{-2i\sigma(\mathrm{w}-\mathrm{y},\mathrm{w}-\mathrm{z})} f(\mathrm{y}) g(\mathrm{z}) \mathrm{dwdydz} \\
&\overset{(d)}{=} \frac{4^d}{(2\pi)^{3d}} \int_\Xi \int_\Xi e^{i\sigma(\mathrm{x}+2\mathrm{y},\mathrm{w})} f(\mathrm{y}) \int_\Xi g(\mathrm{z}) e^{i\sigma(2\mathrm{w}-2\mathrm{y},\mathrm{z})} \mathrm{dzdwdy} \\
&= \frac{4^d}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{i\sigma(\mathrm{x}+2\mathrm{y},\mathrm{w})} f(\mathrm{y}) \mathcal{F}_\Xi g(2\mathrm{w} - 2\mathrm{y}) \mathrm{dwdy} \\
&\overset{(e)}{=} \frac{1}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{i\sigma(\mathrm{x}+2\mathrm{y},\mathrm{w}/2+\mathrm{y})} f(\mathrm{y}) \mathcal{F}_\Xi g(\mathrm{w}) \mathrm{dwdy} \\
&\overset{(f)}{=} \frac{1}{(2\pi)^{2d}} \int_\Xi e^{i\sigma(\mathrm{x},\mathrm{y})} f(\mathrm{y}) \int_\Xi e^{i\sigma(\mathrm{x}/2+\mathrm{y},\mathrm{w})} \mathcal{F}_\Xi g(\mathrm{w}) \mathrm{dwdy} \\
&= \frac{1}{(2\pi)^d} \int_\Xi e^{i\sigma(\mathrm{x},\mathrm{y})} f(\mathrm{y}) g\left(\frac{\mathrm{x}}{2} + \mathrm{y}\right) \mathrm{dy}
\end{aligned}
$$

where (d) and (f) hold because σ is the (standard) symplectic bilinear form while (e) is obtained by changing coordinate w. Now it is much simpler to check the claim:

$$
\begin{aligned}
&(\mathfrak{Op}(f \circ g)u)(z) \\
&= \frac{1}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{i(y\cdot\xi - x\cdot\eta)} f(y,\eta) g\left(\frac{x}{2} + y, \frac{\xi}{2} + \eta\right) e^{-i(x/2+z)\cdot\xi} u(x+z) \mathrm{dxdy} \\
&= \frac{2^d}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{i(2y\cdot\xi - 2y\cdot\eta + 2z\cdot\eta - x\cdot\xi - z\cdot\xi)} f(y,\eta) g\left(\frac{x - z + 2y}{2}, \xi\right) u(x) \mathrm{dxdy} \\
&= \frac{2^d}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} e^{2iz\cdot\eta - 2iy\cdot\eta} f(y,\eta) (\mathfrak{Op}(g)u)(2y - z) \mathrm{d}\eta \mathrm{d}y \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} e^{i(z-y)\cdot\eta} f\left(\frac{y+z}{2}, \eta\right) (\mathfrak{Op}(g)u)(y) \mathrm{d}\eta \mathrm{d}y \\
&= (\mathfrak{Op}(f)\mathfrak{Op}(g)u)(z)
\end{aligned}
$$

Hence, the claim that $\mathfrak{Op}(f)\mathfrak{Op}(g) = \mathfrak{Op}(f \circ g)$ is shown.

$\square$