VI Interval estimation

(linked to hypothesis test)

Interval estimation covers several related notions:

→ • confidence intervals
• credible intervals (Bayes approach)
• likelihood intervals
• tolerance intervals

VI.1 Confidence intervals

Def: A confidence interval for a parameter θ is a pair of statistics

$L(\underline{X}), U(\underline{X})$ with $L(\underline{X}) < U(\underline{X})$ (Lower & Upper)

together with a confidence coefficient γ given by

$\gamma := \inf_{\theta} P(\theta \in [L(\underline{X}), U(\underline{X})])$ we always take the worst scenario.

if there exists additional parameters, we also take the infimum on them

Remarks:

• One can accept that $L(\underline{X}) = -\infty$ and $U(\underline{X}) = +\infty$ but not both.
• Usually γ is denoted by $1-\alpha$ and we speak about a $(1-\alpha)$ confidence.

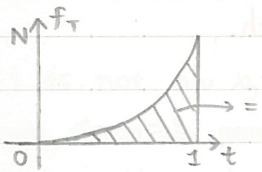
Example:

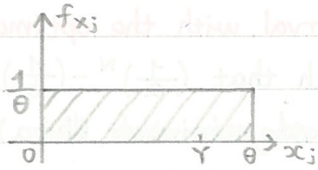
Let $\underline{X} = (X_1, \cdots, X_N)$ with $X_j \sim$ uniform $(0, \theta)$

Let $Y := \max\{X_1, \cdots, X_N\} = X_{(N)}$ last ordered statistics and set $T := \frac{Y}{\theta}$

What is the distribution of T? T has a distribution f_T with

$f_T(t) = Nt^{N-1}$ for $t \in [0,1]$ (exercise based on § II.3)





Let us consider 2 possible confidence coefficients:

1) $[L(\underline{X}), U(\underline{X})] = [aY, bY]$ for $1 \leq a < b$
2) $[L(\underline{X}), U(\underline{X})] = [Y+c, Y+d]$ for $0 \leq c < d$

For 1)

$P(\theta \in [aY, bY]) = P(aY \leq \theta \leq bY) = P(a \leq \frac{\theta}{Y} \leq b)$

$= P(\frac{1}{b} \leq \frac{Y}{\theta} \leq \frac{1}{a}) = \int_{1/b}^{1/a} N t^{N-1} dt = (\frac{1}{a})^N - (\frac{1}{b})^N$

$\inf_\theta P(\theta \in [aY, bY]) = (\frac{1}{a})^N - (\frac{1}{b})^N$ no dependence on θ!

For 2)

$P(\theta \in [Y+c, Y+d]) = P(Y+c \leq \theta \leq Y+d)$

$= P(1-\frac{d}{\theta} \leq \frac{Y}{\theta} \leq 1-\frac{c}{\theta}) = \int_{1-d/\theta}^{1-c/\theta} N t^{N-1} dt = (1-\frac{c}{\theta})^N - (1-\frac{d}{\theta})^N$

$\inf_\theta P(\theta \in [Y+c, Y+d]) = \inf_\theta \left((1-\frac{c}{\theta})^N - (1-\frac{d}{\theta})^N\right) = 0$

The 2nd choice is not really good

since for any c and d, the confident coefficience is 0.

For 1), if we impose that

$(\frac{1}{a})^N - (\frac{1}{b})^N = 1-\alpha$ for a given α,

we can find some a and b.

no uniqueness: different possible choices for (a, b)

Def. A random variable $Q(\underline{X}, \theta)$ this is not a statistic (explicit dependence on θ) is called a **pivotal quantity** or a **pivot** if its distribution function does not depend on θ.

Exemple: $\frac{Y}{\theta}$

Def. Among all statistics $L(\underline{X})$, $U(\underline{X})$ satisfying (fixed)

$L(\underline{X}) < U(\underline{X})$ and $\inf_\theta P(\theta \in [L(\underline{X}), U(\underline{X})]) = 1-\alpha$

the ones having the shortest length $U(\underline{X}) - L(\underline{X})$ define the **$(1-\alpha)$-confidence interval with the optimal length**.

Exercise: find a and b such that

$(\frac{1}{a})^N - (\frac{1}{b})^N = 1-\alpha$ and minimum $(b-a)$? among $1 \leq a < b$

VI.2 Relation with hypothesis test

Example: Suppose $X_j \sim n(\mu, \sigma^2)$ and $H_0: \mu = \mu_0$ (given)

Test: rejecting H_0 if $\left|\underbrace{\frac{\bar{X}-\mu}{\sigma/\sqrt{N}}}_{=: Z \sim n(0,1)}\right| = |Z| \geq Z_{\alpha/2}$ (σ known) with $P(|Z| \geq Z_{\alpha/2}) = \frac{\alpha}{2}$

$\Rightarrow$ This test is an α-level test.

$\Leftrightarrow$ Accept H_0 if $\left|\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{N}}\right| < Z_{\alpha/2}$

$\Leftrightarrow \quad -Z_{\alpha/2} < \dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{N}} < Z_{\alpha/2}$

$\Leftrightarrow \bar{X} - Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}} < \mu_0 < \bar{X} + Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}}$ with

$P\left(\bar{X} - Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}} < \mu_0 < \bar{X} + Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}} \,\middle|\, \mu = \mu_0\right) \overset{\text{true for any } \mu_0}{=} 1-\alpha$

$\Leftrightarrow P\left(\bar{X} - Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}} < \mu < \bar{X} + Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}}\right) = 1-\alpha$

$\Leftrightarrow P\left(\mu \in \left[\bar{X} - Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}}, \bar{X} + Z_{\alpha/2}\dfrac{\sigma}{\sqrt{N}}\right]\right) = 1-\alpha$

Thus, we have obtained a $(1-\alpha)$ confidence interval from a hypothesis test.

More generally:

Thm. For any $\theta \in \overset{\subset \mathbb{R}}{\Theta}$ parameter space set $H_0 : \theta = \theta_0$ and consider a level α-test $\mathcal{T}_{\theta_0}$ for H_0.

Let $\mathcal{A}(\theta_0) = \{\underline{x} \mid \mathcal{T}_{\theta_0}(\underline{x}) = H_0 \text{ accepted}\}$ be the acceptance region of this test (the complement of the rejection region).

For any $\underline{x}$, set

$C(\underline{x}) := \{\theta \in \Theta \mid \underline{x} \in \mathcal{A}(\theta)\}$

Then $C(\underline{x})$ is a $(1-\alpha)$ confidence set.

[Thm. 9.2.2] proof not long

Remark:

1) $C(\underline{x})$ is not always an interval (but there are conditions such that one gets an interval)

2) It is possible to look for an optimal balance between the confident coefficient and the length of $C(\underline{X})$.