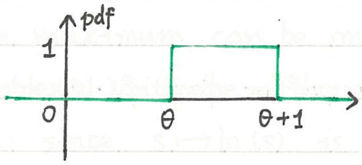
Def. A statistic $T(\underline{X})$ whose pdf or pmf does not depend on θ is called an ancillary statistic.

Example: Let $\underline{X} = (X_1, \cdots, X_N)$ a random sample with $X_j \sim \text{unif}(\theta, \theta+1)$ with



$$f_{X_j}(x) = \begin{cases} 1 & \text{if } x \in (\theta, \theta+1) \\ 0 & \text{otherwise} \end{cases}$$

Then we set $R(\underline{X}) \underset{\text{statistic}}{\overset{\text{range}}{}} = X_{(N)} - X_{(1)}$ (maximal stat.) (minimal stat.)

Then the pdf of $R(\underline{X})$ is the function

$x \longmapsto N(N-1)x^{N-2}(1-x)$ for $x \in (0,1)$ exercise: someone could show this.

which is θ-independent.

$\Rightarrow R(\underline{X})$ is an ancillary statistic.

⚠ A sufficient statistic can be related to an ancillary statistic.

Example: in the previous example,

$(R(\underline{X}), M(\underline{X}) := \frac{1}{2}(X_{(1)} + X_{(N)}))$ is a sufficient statistic for θ.

It means that both information are necessary,

and $R(\underline{X})$ alone does not ~~s~~ say anything on θ.

III.2 Likelihood principle

Def. Let $\underline{X} := (X_1, \cdots, X_N)$ be a r. sample with joint pdf or pmf $f_{\underline{X}}(\cdot|\theta)$.

For a given observation $\underline{x}$, the function

$$\theta \longmapsto L(\theta|\underline{x}) := f_{\underline{X}}(\underline{x}|\theta)$$

is called the likelihood function at $\underline{x}$.

Likelihood principle

If $\underline{x}$ and $\underline{y}$ satisfy $L(\theta|\underline{x}) = cL(\theta|\underline{y})$ $\forall\theta$ and a fixed c,

then the inference on θ from $\underline{x}$ and $\underline{y}$ should be the same.

Example: For $\underline{X} = (X_1, \cdots, X_N)$ with $X_j \sim n(\mu, \sigma^2)$, one has

$$f_{\underline{X}}(\underline{x}|\mu,\sigma^2) = e^{-N(\bar{x}-\mu)^2/2\sigma^2}(2\pi\sigma^2)^{-N/2}e^{-\sum_j (x_j-\bar{x})^2/2\sigma^2}$$

For $\theta = \mu$, one has

$$L(\theta|\underline{x}) = cL(\theta|\underline{y}) \text{ if } \bar{x} = \bar{y} \text{ and } \underset{\text{indep of } \theta}{c} = \exp\left(\sum_j \frac{-(x_j-\bar{x})^2 + (y_j-\bar{y})^2}{2\sigma^2}\right)$$

Thus from this principle, if $\bar{x} = \bar{y}$ the inference on θ is same from $\underline{x}$ or $\underline{y}$.