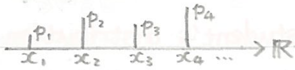
Computation related to $X_{(j)}$

Prop. (discrete case)

Suppose $X_1, \cdots, X_N$ are iid with X_j discrete with

$\mathrm{Ran}(X_j) = \{x_i\}_{i \in \mathbb{N}}$



and s.t. $x_i \leq x_{i+1}$ and $f_{X_j}(x_i) =: p_i$ with $\sum_{i \in \mathbb{N}} p_i = 1$. Then

$$P(X_{(j)} \leq x_i) = \sum_{k=j}^{N} \binom{N}{k} P_i^k (1-P_i)^{N-k} \quad \text{with}$$

$$P_i = F_{X_j}(x_i) = P(X_j \leq x_i) = \sum_{k=1}^{i} p_k$$

Proof

Set $Y := \overset{\text{number}}{\#}\{X_j \leq x_i \text{ for } j = 1, \cdots, N\}$ number of X_j taking a value smaller than x_i

But $\{X_j\}$ are i.i.d and $P(X_j \leq x_i) = P_i$

Then $Y \sim \text{binomial}(N, P_i)$.

Then $\{X_{(j)} \leq x_i\} = \{Y \geq j\}$ key point (both are subsets of $S \times \cdots \times S$)

$$\Rightarrow \underbrace{P(X_{(j)} \leq x_i)}_{\text{want to compute}} = P(Y \geq j) \quad \text{because of binomial}$$

$$= \sum_{k=j}^{N} P(Y=k) = \sum_{k=j}^{N} \binom{N}{k} P_i^k (1-P_i)^{N-k} \quad (*)$$

□

Prop. (continuous case) (Thm. 5.4.4)

If X_j has cdf F and pdf f, then

$$f_{X_{(j)}}(x) = \frac{N!}{(j-1)!(N-j)!} f(x) F(x)^{j-1} (1-F(x))^{N-j}$$

pdf for having j r.v. X_k taking a value smaller than x.

II.4 Computing with random samples

Aim: compute $(*)$ numerically with random samples.

Idea: use the weak law of large number (from Appendix 3).

Consider $\underline{X}$ of i.i.d. r.v. with $E(X_j) = \mu$ and $\mathrm{Var}(X_j) = \sigma^2$

Then $\overline{X_N} = \frac{1}{N} \sum_{j=1}^{N} X_j$ (sample mean);

$\overline{X_N} \xrightarrow{N \to \infty} \mu$ in probability

$\Leftrightarrow \forall \varepsilon > 0,\ P(|\overline{X_N} - \mu| \geq \varepsilon) \xrightarrow{N \to \infty} 0$

1) Consider $X_1, \cdots, X_N$ discrete r.v. satisfying the prescribed distribution of the previous propersition.

2) Set $Y_k := \begin{cases} 1 & \text{if at least } j\ X_\ell \text{ take a value} \leq x_i \\ 0 & \text{otherwise} \end{cases}$

3) Observe that $Y_k \sim \text{Bernoulli}(q)$ with $q = P(X_{(j)} \leq x_i)$, and $E(Y_k) = q$

4) Then $\{Y_k\}$ by repeating the experiment is an i.i.d family with $E(Y_k) = q$.

By weak law of large number, $Y \xrightarrow[\text{experiments}]{\text{for many}} q$

What is missing: how to generate random numbers following a prescribed distributio

There are solutions. See section 5.6 (exercise : report on this)