Exercise: Summary ~~t~~ one distribution in the handout for this lecture, writing the properties and an example for this distribution. (do in pairs)

Remark: From F_X one gets

- $P(a < X \leq b) = P(\{\omega \in S \mid X(\omega) \in (a,b]\}) = F_X(b) - F_X(a)$
- Since $P(X<a) = \lim_{\varepsilon \searrow 0} F_X(a-\varepsilon)$

$\Rightarrow P(a \leq X \leq b) = F_X(b) - \lim_{\varepsilon \searrow 0} F_X(a-\varepsilon)$

$\Rightarrow \underbrace{P(X=a)}_{\text{non-zero whenever } F_X \text{ has a jump at } a} = F_X(a) - \lim_{\varepsilon \searrow 0} F_X(a-\varepsilon)$ see the figure in p. 3

Def.

1) X is CONTINUOUS if "F_X has no jump", or

$\lim_{\varepsilon \searrow 0} F_X(a-\varepsilon) = F_X(a) \ \forall a \in \mathbb{R}$

2) X is (ABSOLUTELY) CONTINUOUS if

$\exists f_X : \mathbb{R} \mapsto \mathbb{R}$ such that $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$; $f_X \in L^1(\mathbb{R})$

PROBABILITY DENSITY FUNCTION (PDF)

3) X is DISCRETE if $X(S) \subset \mathbb{R}$ is finite or countable.
$\Updownarrow$
(in bijection with $\mathbb{N}$)

4) X can be SINGULAR CONTINUOUS,

or a mixture of abs. continuous, sing. continuous and discrete.

Remark

When X is discrete, the function

$f_X : \mathbb{R} \mapsto \mathbb{R}$, $f_X(x) := F_X(x) - \lim_{\varepsilon \searrow 0} F_X(x-\varepsilon)$ is called the PROBABILITY MASS FUNCTION (pmf).

Observe that $\sum_{x \in \mathbb{R}} f_X(x) = 1$.

And for the pdf $\int_{-\infty}^{\infty} f_X(y)\,dy = 1$.

Use of pdf and pmf:

If $I \subset \mathbb{R}$ (Borel subset of $\mathbb{R}$), then

$$P(x \in I) = P(\{x \in S \mid X(x) \in I\}) = \begin{cases} \int_I f_X(y)\,dy & \text{in continuous case} \\ \sum_{x \in I} f_X(x) & \text{in discrete case} \end{cases}$$

Remark:

The set $\{X(s) \mid s \in S\}$ is called the IMAGE of the random variable X.

We denote it by $Im(X)$ or $Ran(X)$

Remark:

If a pdf or a pmf depends on some parameters $\theta_1, \theta_2, \cdots, \theta_n$, then we write

$f(\cdot \mid \theta_1, \cdots, \theta_n) : \mathbb{R} \longmapsto \mathbb{R}$

Remark:

The range of a discrete r.v. is countable, while

the range of a countinuous one is not countable.

I.3 Transformations and expectations

We always use $(S, \mathcal{B}, P)$ for a prob. space and X for a random variable.

Let $g : \mathbb{R} \mapsto \mathbb{R}$ and consider $g(X) := g \circ X$

$$\begin{tikzcd} S \arrow[r, "X"] \arrow[rr, bend right, "g \circ X"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

Lemma: $g(X)$ is a random variable if $g : \mathbb{R} \mapsto \mathbb{R}$ is BOREL MEASURABLE.

It means that $\forall A \in \sigma_B : g^{-1}(A) \in \sigma_B$

Borel algebra on $\mathbb{R}$

Remark: This is a condition satisfied by almost all functions

(for example continuous, or piecewise continuous functions).

Def. If X is discrete or (absolutely) continuous with pmf or pdf f,

then the EXPECTED VALUE or MEAN of X is given by

$E(X) := \int_{-\infty}^{\infty} x f_X(x)\,dx$ or $\sum_x x f(x)$

if it converges absolutely

$(:\Leftrightarrow \int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$ or $\sum_x |x| f(x) < \infty)$

← Exercise: find an example when E(X) does not exist.

Thm. If X is continuous or discrete and g Borel measurable, then

$E(g(X)) = \int_{\mathbb{R}} g(x) f_X(x)\,dx$

whenever it converges absolutely.

Def. $Var(X) = E\left((X - E(X))^2\right)$

is the VARIANCE whenever it exists.

$\sqrt{Var(X)}$ is called the STANDARD DERIVATION.

Remark: If $E(X)=\mu$ and $Var(X)=\sigma^2$, then by setting

$$Z := \frac{X-\mu}{\sigma} \text{ (the STANDARD FORM)}$$

one has $E(Z)=0$ and $Var(Z)=1$.

Def. The k^{th}-moment of X are defined by $E(X^k)$, and the central k^{th}-moment of X by $E((X-E(X))^k)$ whenever they exist.

Also $M_X(t) := E(e^{tX})$ for $t \in \mathbb{R}$ is called the moment generating function (mgf).

I.4 Multiple random variables

We are going to consider n random variables $X_1, X_2, \cdots, X_N : S \mapsto \mathbb{R}$

$$\Leftrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} =: \underline{X} : S \mapsto \mathbb{R}^N$$

Def. The joint cumulative distribution function associated with $\underline{X}$ is a function

$F_{\underline{X}} : \mathbb{R}^N \longmapsto [0,1]$ defined for $\underline{x} = (x_1, \cdots, x_n)$ by

$$\begin{aligned} F_{\underline{X}}(x) &:= F_{X_1,\cdots,X_N}(x_1, \cdots, x_n) \\ &:= P(\{s \in S \mid X_1(s) \le x_1, X_2(s) \le x_2, \cdots, X_n(s) \le x_n\}) \\ &= P(X_j \le x_j \ \forall j \in \{1, \cdots, N\}) \end{aligned}$$

If $\underline{X}$ takes only a countable number of values in $\mathbb{R}^N$, the joint probability mass function is defined by

$$f_{\underline{X}}(x_1, \cdots, x_N) = P(X_1 = x_1, \cdots, X_N = x_N)$$

$\underline{X}$ is an absolutely continuous random vector if

$$F_{\underline{X}}(x_1, \cdots, x_N) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_N} \underset{\uparrow}{f_{\underline{X}}}(y_1, y_2, \cdots, y_N)\, dy_1\, dy_2 \cdots dy_N$$

joint probability density function

If $A \subset \mathbb{R}^N$ (Borel subset) then

$$P(X \in A) = \iint_A \cdots \int f_X(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$$

Def. $X_1, \cdots, X_N$ are independent r.v. if

$$P(X_1 \le x_1, X_2 \le x_2, \cdots, X_N \le x_n) = P(X_1 \le x_1) P(X_2 \le x_2) \cdots P(X_N \le x_n)$$

$$\Updownarrow$$

$$F_{\underline{X}}(x_1, \cdots, x_n) = \prod_{j=1}^{N} F_{X_j}(x_j)$$

$\Updownarrow$ ← only in the case of discrete or abs. continuous

$$f_{\underline{X}}(x_1, \cdots, x_n) = \prod_{j=1}^{N} f_{X_j}(x_j)$$