

M²

Ex 1.1.7 (a) Let $\{f_n\}$ be a strong Cauchy sequence

By Lemma 1.1.5, $\{f_n\}$ is also weak Cauchy and it suffices to show that the limit of a weak Cauchy sequence is unique.

assume that $w\text{-}\lim f_n = h_1$ and $w\text{-}\lim f_n = h_2$ where $h_1, h_2 \in \mathbb{H}$

Then, for any $g \in \mathbb{H}$

$$\begin{aligned} \langle g, h_1 - h_2 \rangle &= \langle g, h_1 \rangle - \langle g, h_2 \rangle = \lim_{n \rightarrow \infty} \langle g, f_n \rangle - \lim_{n \rightarrow \infty} \langle g, f_n \rangle \\ &= \lim_{n \rightarrow \infty} (\langle g, f_n \rangle - \langle g, f_n \rangle) = 0 \end{aligned}$$

we take $g = h_1 - h_2$, then

$$\Rightarrow \|h_1 - h_2\|^2 = 0 \Rightarrow h_1 = h_2$$

(b) the boundedness of the limit of a strong Cauchy sequence:

Let $\{f_n\}$ be a strong Cauchy sequence and $s\text{-}\lim f_n = f$

then by Lemma 1.1.5 $\lim_{n \rightarrow \infty} \|f_n\| = \|f\|$

since $\|f_n\| < \infty, \forall n$

$$\Rightarrow \|f\| < \infty$$

the boundedness of the limit of a weak Cauchy sequence

Let $\{f_n\}$ be a weak Cauchy sequence and $w\text{-}\lim f_n = f$ i.e. $\lim_{n \rightarrow \infty} \langle g, f_n \rangle = 0, \forall g \in \mathbb{H}$

Define $\varphi_n: \mathbb{H} \rightarrow [0, +\infty)$

$$g \mapsto |\langle f_n, g \rangle|$$

then

$$\begin{aligned} \varphi_n(f+g) &= |\langle f_n, f+g \rangle| = |\langle f_n, f \rangle + \langle f_n, g \rangle| \leq |\langle f_n, f \rangle| + |\langle f_n, g \rangle| \\ &= \varphi_n(f) + \varphi_n(g) \end{aligned}$$

and for any fixed $h \in \mathbb{H}$

$\{\varphi_n(h)\}_{n \in \mathbb{N}}$ is bounded, since

$$\varphi_n(h) = |\langle f_n, h \rangle| \rightarrow |\langle f, h \rangle| \text{ as } n \rightarrow \infty$$

So by Thm 1.1.8 (Uniform boundedness principle)

$\exists M > 0, \text{ s.t.}$

$$\text{we } \sup_n \varphi_n(u) \leq M, \forall u \in \mathbb{H} \text{ and } \|u\|=1$$

then consider $|\langle u, f_n \rangle|$,

$$|\langle u, f_n \rangle| \leq \|u\| \cdot \|f_n\| = \|f_n\|, \forall u \in \mathbb{H} \text{ and } \|u\|=1$$

we take $u = f_n/\|f_n\|$, then

$$\Rightarrow |\langle u, f_n \rangle| = \|f_n\|$$

It means that $\|f_n\| = \sup_{\substack{\|u\|=1 \\ u \in \mathbb{H}}} |\langle u, f_n \rangle|$

$$\therefore \|f_n\| = \sup_{\substack{\|u\|=1 \\ u \in \mathbb{H}}} |\langle u, f_n \rangle| = \sup_{\substack{\|u\|=1 \\ u \in \mathbb{H}}} \varphi_n(u) \leq M, \text{ for } \forall n \in \mathbb{N}.$$

$$\Rightarrow \|f\| = \sup_{\substack{\|u\|=1 \\ u \in \mathbb{H}}} |\langle u, f \rangle| = \sup_{\substack{\|u\|=1 \\ u \in \mathbb{H}}} \lim_{n \rightarrow \infty} |\langle u, f_n \rangle| < +\infty.$$

assume that

Ex 1.1.12. (\Rightarrow) M is dense in H and let $f \in M^\perp$

since $f \in H = \overline{M}$

\exists cauchy sequence $\{f_n\} \subset M$, s.t.

$$s\text{-}\lim f_n = f$$

Hence,

$$|\langle f, f \rangle| \leq |\langle f, f - f_n \rangle| + |\langle f, f_n \rangle| = |\langle f, f - f_n \rangle| \leq \|f\| \|f - f_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow f = 0$$

(\Leftarrow). assume that $M^\perp = \{0\}$

Let T be the subspace spanned by M .

we first show that the following two statements is equivalent: for $f \in T$

(i) $f \perp M$

(ii) $f \perp T$

(ii) \Rightarrow (i) is trivial.

(i) \Rightarrow (ii) for $\forall t \in T$, t is of the form $t = \sum_{k=1}^N \alpha_k m_k$ with $m_k \in M$ for some $N < \infty$. hence

$$\langle f, t \rangle = \langle f, \sum_{k=1}^N \alpha_k m_k \rangle = \sum_{k=1}^N \alpha_k \langle f, m_k \rangle = 0$$

Hence $f \perp T$

So, by $M^\perp = \{0\}$, we can know that $T^\perp = \{0\}$

by 'Projection' theorem

$$T^\perp = \{0\}^\perp = H$$

$$\Rightarrow H = T \subset \overline{M} \subset H \Rightarrow \overline{M} = M$$

Hence M is dense in H

Prop. 1.4.6. (vi). if $g \perp \text{Ran}(v)$

then $0 = \langle g, vf \rangle = \langle v^* g, f \rangle$, $\forall f \in H$.

$$\Rightarrow v^* g \in H^\perp = \{0\} \Rightarrow v^* g = 0$$

Exercise 3.1.4. Show that the following relations hold on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$:

$$[iX_j, X_k] = 0 = [D_j, D_k] \quad \text{for any } j, k \in \{1, \dots, d\}$$

$$\text{while } [iD_j, X_k] = 1 \delta_{jk}$$

pf. for any $f \in \mathcal{S}(\mathbb{R}^d)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$\begin{aligned} ([iX_j, X_k] f)(x) &= (iX_j X_k f)(x) - (iX_k X_j f)(x) \\ &= ix_j x_k f(x) - ix_k x_j f(x) = 0 \end{aligned}$$

Hence $[iX_j, X_k] = 0$

$$\begin{aligned} [D_j, D_k] f &= D_j D_k f - D_k D_j f \\ &= \mathcal{F}^* X_j \mathcal{F}^{-1} X_k \mathcal{F} f - \mathcal{F}^* X_k \mathcal{F}^{-1} X_j \mathcal{F} f \\ &= \mathcal{F}^* [X_j, X_k] \mathcal{F} f \\ &= 0 \end{aligned}$$

Hence $[D_j, D_k] = 0$

since $D_j = -i \partial_j$

$$([iD_j, X_k] f)(x) = \partial_j (X_k f(x)) - X_k \partial_j f(x) = 0, \quad \text{when } j \neq k$$

$$\begin{aligned} ([iD_j, X_j] f)(x) &= \partial_j (x_j f(x)) - x_j \partial_j f(x) = f(x) + x_j \partial_j f(x) - x_j \partial_j f(x) \\ &= f(x) \end{aligned}$$

Hence

$$[iD_j, X_j] = 1$$

Hence

$$[iD_j, X_k] = i \delta_{jk}$$