

Ex. 2.3.2  $f \in D(A) \subset \mathcal{H}$ 

$$i) \|Bf\|^2 \leq a^2 \|Af\|^2 + b^2 \|f\|^2 \quad (\because \circ) = a^2 (\|Af\|^2 + \frac{b^2}{a^2} \|f\|^2)$$

$$\begin{aligned} \|(A \pm ic)f\|^2 &= \langle (A \pm ic)f | (A \pm ic)f \rangle = \|Af\|^2 + c^2 \|f\|^2 \\ &\quad + ic \langle Af, f \rangle - ic \langle f, Af \rangle = \|Af\|^2 + c^2 \|f\|^2 \end{aligned}$$

(\(\because A\) is self-adjoint op.)

$$\text{Then } \|Bf\|^2 \leq a^2 \|(A \pm ic)f\|^2 \quad \text{where } c = \frac{b}{a}$$

Because  $A$  is self-adjoint,  $\pm i \frac{b}{a} \in \mathbb{C}$  belong to  $\rho(A)$ .

$$\text{Therefore, } \|B(A \pm i \frac{b}{a})^{-1}f\| \leq a \|f\|, \quad \forall f \in \mathcal{H}$$

$$\text{It can be seen as } \|B(A \pm i \frac{b}{a})^{-1}\| \leq a$$

$$ii) B(A-z)^{-1} \in \mathcal{B}(\mathcal{H}) \text{ where } z \in \rho(A)$$

$$\begin{aligned} \|Bf\| &= \|B(A-z)^{-1}(A-z)f\| \leq \|B(A-z)^{-1}\| \|(A-z)f\| \\ &\leq \|B(A-z)^{-1}\| (\|Af\| + |z| \|f\|) \end{aligned}$$

$$\therefore B \text{ is } A\text{-bounded with } \alpha = \|B(A-z)^{-1}\|$$

iii) From ii)  $\alpha = \|B(A-z)^{-1}\|$  is a finite number,  
 $\therefore B(A-z)^{-1}$  is bounded.

$$A\text{-bound } b \leq \inf_{z \in \rho(A)} \|B(A-z)^{-1}\| \quad \therefore$$

To show that this inequality is in fact equality, we show that when  $\delta > 0$  is given,  $\exists K \in \mathbb{R}$  s.t.

$$\|B(A \pm iK)^{-1}\| \leq b + \delta$$

$\|Bf\| \leq \alpha \|Af\| + \beta \|f\|$  is satisfied for  $\alpha > b$  then

$$\alpha = b + \frac{\delta}{2}, \quad \delta > 0,$$

$a$  in  $\mathbb{D}$  satisfies  $a > \alpha$  then  $a = b + \delta$

$$\text{Therefore from i) } \|B(A \pm i \frac{b}{b+\delta})\| \leq b + \delta \quad \text{''}$$

where  $K = \frac{b}{b+\delta}$

Ex. 3.1.5

We study 1-dim harmonic oscillators.

Hamiltonian of this system is

$$H = \sum_{j=1}^d D_j^2 + 20^2 X_j^2 = \sum_{j=1}^d H_j, \quad D(H) = S(\mathbb{R}^d)$$

where  $D_j = -i\partial_{x_j}$ ,  $d =$  number of oscillators (they're not correlated)

$D_j, X_j$  fulfill the below commutation relation on  $S(\mathbb{R}^d)$

$$[D_j, X_j] = -i$$

Now we define their linear combinations as follows:

$$a_j^+ := \omega X_j - i D_j, \quad a_j := \omega X_j + i D_j \quad \text{on } S(\mathbb{R}^d)$$

They fulfill the below relations:

$$[a_j, a_j^+] = 2\omega, \quad [a_i, a_j] = [a_i^+, a_j^+] = 0 \quad \forall i, j$$

We introduce the operator  $N_j := a_j^+ a_j \quad \forall j$

They fulfill the relations on  $S(\mathbb{R}^d)$ :

$$[N_j, a_j^+] = 2\omega a_j^+$$

$$[N_j, a_j] = -2\omega a_j \quad \forall j$$

$$H_j = N_j + \omega \text{ lead to } H = \sum_{j=1}^d N_j + d\omega$$

$D_j, X_j$  are essentially self-adjoint, so  $(a_j^+)^+ = a_j$ . Then  $N_j$  is essentially self-adjoint.

Then it's eigenstates associated to different eigenvalues are orthogonal.

$$\int e^{-wx^2} dx = \sqrt{\frac{\pi}{w}}$$

No.

Date

$$|0\rangle_j = \left(\frac{w}{\pi}\right)^{\frac{1}{4}} e^{-\frac{w}{2}x_j^2}, \quad |2w\rangle_j = \frac{1}{\sqrt{2w}} a_j^\dagger |0\rangle_j, \quad |4w\rangle_j = \frac{1}{\sqrt{2}(\sqrt{2w})^2} a_j^{\dagger 2} |0\rangle_j,$$

$$\dots, \quad |2nw\rangle_j = \frac{1}{\sqrt{n!}} \frac{1}{(\sqrt{2w})^n} a_j^{\dagger n} |0\rangle_j, \dots$$

$$\partial_x e^{-x^2} = -2x e^{-x^2}, \quad \partial_x^2 e^{-x^2} = -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$(\partial_x^2 - 4x^2) e^{-x^2} = -2e^{-x^2} \text{ it says } e^{-x^2} \text{ is stable under } \partial_x^2 - x^2$$

Continuing this calculation, one finds  $e^{-x^2} \in S(\mathbb{R})$

These states are generated by  $a_j^\dagger$ , so  $a^\dagger$  should be called creation operator.

Considering system is oscillators not correlating each other.

And each oscillator is isolated from the other ones.

Then eigenvalue of  $\mathcal{H}$  associated with state of an oscillator is constant.

Then state of oscillator is described as linear combination with respect to the orthonormal basis.

This is from the fact that  $\mathcal{H}$  acts on  $S(\mathbb{R}^d)$  linearly.

i.e. eigenstate of  $\mathcal{H}$  is written by

$$|\psi\rangle = \bigotimes_{j=1}^d \sum_n c_n^j |2nw\rangle_j$$