Nagoya University, G30 program Spring 2019
Calculus II: Final examination Instructor : Serge Richard

Exercise 1 *Consider the parametric curve in* $\mathbb{R}^3$ *defined by* $[0,1] \ni t \mapsto \big(\mathrm{e}^t, \sqrt{2}\,t, \mathrm{e}^{-t}\big) \in \mathbb{R}^3$ *and compute the length of this curve (it could be useful to remember that* $\cosh(t) = (\mathrm{e}^t + \mathrm{e}^{-t})/2$ *).*

Exercise 2 *Consider the map*

$$f : \mathbb{R}^2 \ni \ (x,y) \mapsto x^4 - 2xy + 2y^2 - 1 \ \in \mathbb{R}.$$

(i) Show that the implicit function theorem can be applied at the point $(1,1) \in \mathbb{R}^2$,

(ii) Compute the tangent at the point $(1,1)$ *of the curve of equation* $f(x,y) = 0$, *and determine the position of this curve with respect to the tangent line at this point.*

Exercise 3 *Compute:*

(i) the curve integral of $f : \mathbb{R}^3 \ni (x,y,z) \mapsto (x, zy, xz-y) \in \mathbb{R}^3$ *along the curve defined by the segment between* $(0,0,0)$ *and* $(1,2,3)$,

(ii) the curve integral of $f : \mathbb{R}^2 \ni (x,y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y) \in \mathbb{R}^2$ *along the curve defined by the parabola of equation* $y = x^2$ *from the point* $(0,0)$ *to the point* $(1,1)$,

(iii) the integral $\iint_\Omega (x-y)\,\mathrm{d}x\,\mathrm{d}y$ *with* Ω *the subset of* $\mathbb{R}^2$ *defined by the three lines of equation* $x = 0$, $y = x+2$, *and* $y = -x$,

(iv) the integral $\iiint_\Omega z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *with* Ω *the interior of the upper half of the ball centered at* 0 *and of radius* 1, *namely* $\Omega = \big\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \text{ and } z \geq 0\big\}$.

Exercise 4 *Consider the* right *half-sphere centered at the origin and of radius* 2 *(*$y \geq 0$*). Consider also the function* $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $\Psi(x,y,z) = (x, z, -y)$. *State Stokes' theorem and verify its validity on this example.*

Exercise 5 *The aim of this exercise is to show that the area of a surface is independent of its parametrization. Let* Ω *be a subset of* $\mathbb{R}^2$ *and let* $f : \Omega \to \mathbb{R}^3$ *be a parametric surface of class* C^1, *namely*

$$\Omega \ni (x,y) \mapsto f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix} \in \mathbb{R}^3.$$

Let also $\mathcal{D} \subset \mathbb{R}^2$ *and* $\phi : \mathcal{D} \to \Omega$, *be a diffeomorphism of class* C^1, *namely*

$$\mathcal{D} \ni (s,t) \mapsto \phi(s,t) = \begin{pmatrix} \phi_1(s,t) \\ \phi_2(s,t) \end{pmatrix} \in \Omega$$

is bijective, continuously differentiable with a continuously differentiable inverse.

(i) Write the expression for the area of $f(\Omega)$,

(ii) Compute $\frac{\partial}{\partial_s}\big(f(\phi(s,t))\big)$ *and* $\frac{\partial}{\partial_t}\big(f(\phi(s,t))\big)$, *and* $\frac{\partial}{\partial_s}\big(f(\phi(s,t))\big) \times \frac{\partial}{\partial_t}\big(f(\phi(s,t))\big)$,

(iii) With the expressions obtained above, write the expression for the area of the surface $f\big(\phi(\mathcal{D})\big)$ *and compare it with the expression for the area of* $f(\Omega)$.

Final examination

Total : 26 pts

Exercise 1 3 pts

$$c(t) = (e^t, \sqrt{2}\, t, e^{-t})$$

$$c'(t) = (e^t, \sqrt{2}, -e^{-t}) \quad 1$$

$$\|c'(t)\| = (e^{2t} + 2 + e^{-2t})^{1/2}$$

$$= ((e^t + e^{-t})^2)^{1/2}$$

$$= (4 \cosh(t)^2)^{1/2}$$

$$= 2 \cosh(t). \quad 1$$

Thus, $L = \int_0^1 \|c'(t)\|\, dt = 2 \int_0^1 \cosh(t)\, dt$

$= 2 \sinh(t)\Big|_0^1 = \underline{\underline{2 \sinh(1).}}$ 1

Exercise 2 5 pts

i) f is clearly of class C^k, for any $k \in \mathbb{N}$, and $\partial_y f(1,1) = -2x + 4y\big|_{x=y=1} = 2 \neq 0$, and $f(1,1) = 0$.
The implicit function theorem can be aplied at this point.
$\Rightarrow \exists\ \varepsilon > 0$ and $\phi : (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ with $\phi(1) = 1$ and of class C^k s.t. $f(x, \phi(x)) = 0 \quad \forall\, x \in (1-\varepsilon, 1+\varepsilon)$. 1

ii) By i) $\frac{d}{dx} f(x, \phi(x)) = 0 \quad \forall\, x \in (1-\varepsilon, 1+\varepsilon)$

$$\Rightarrow \quad 4x^3 - 2\phi(x) - 2x\phi'(x) + 4\phi(x)\phi'(x) = 0$$

$$\Leftrightarrow \quad 4x^3 - 2\phi(x) = \phi'(x)\big(2x - 4\phi(x)\big)$$

$$\Leftrightarrow \quad \phi'(x) = \frac{4x^3 - 2\phi(x)}{2x - 4\phi(x)} \quad 1$$

and in particular for $x = 1$, $\phi'(1) = \frac{4-2}{2-4} = \frac{2}{-2} = \underline{\underline{-1.}}$ 1

The equation of the tangent is $(y-1) = -1(x-1)$

$\Leftrightarrow \underline{\underline{y = -x + 2}}$. 1

the tangent go through $(1,1)$.

The position is determined by $\phi''(1)$. For this one has:

$$\phi''(x) = \frac{(12x^2 - 2\phi'(x))(2x - 4\phi(x)) - (2 - 4\phi'(x))(4x^3 - 2\phi(x))}{(2x - 4\phi(x))^2}$$

which can be evaluated at $x = 1$, $\phi(1) = 1$, $\phi'(1) = -1$:

$$\phi''(1) = \frac{(12+2)(2-4) - (2+4)(4-2)}{(2-4)^2} = \frac{-14 \cdot 2 - 6 \cdot 2}{4} = -\frac{40}{4} = \underline{-10}.$$

The function is concave at $(1,1)$ $\Rightarrow$ the tangent is above the curve ($\Leftrightarrow$ the curve is below the tangent). 1

Exercise 3

8 pts

i) $c(t) = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$, $c'(t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ for $t \in [0,1]$

$$\int_0^1 f(c(t)) \cdot c'(t)\, dt = \int_0^1 \begin{pmatrix} t \\ 6t^2 \\ 3t^2 - 2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} dt$$

$$= \int_0^1 (t + 12t^2 + 9t^2 - 6t)\, dt = \int_0^1 (21t^2 - 5t)$$

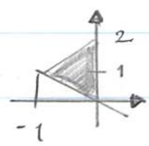
$$= \frac{21}{3} t^3 \Big|_0^1 - \frac{5}{2} t^2 \Big|_0^1 = 7 - \frac{5}{2} = \underline{\underline{\frac{9}{2}}}.$$ 2

ii) f is the gradient of a potential function ϕ with

$$\phi(x,y) = x^3 y + x^2 + xy^3 - y^2 + \text{cst}$$

$$\Rightarrow \int_c f = \phi(1,1) - \phi(0,0) = 2 - 0 = \underline{\underline{2}}.$$ 2

iii) $$\iint_\Omega (x-y)\, dx\, dy = \int_{-1}^0 \left[\int_{-x}^{x+2} (x-y)\, dy \right] dx$$



$$= \int_{-1}^0 \left(xy - \frac{1}{2} y^2 \right)_{y=-x}^{y=x+2} dx$$

$$= \dots = \int_{-1}^2 (2x^2 - 2)\, dx = \frac{2}{3} x^3 - 2x \Big|_{-1}^0 = \underline{\underline{-\frac{4}{3}}}.$$ 2

iv) Since the ball is described by

$$(r,\theta,\varphi) \longmapsto \begin{pmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix} \text{ in spherical coordinates}$$

one has $\iiint_\Omega z \, dx \, dy \, dz$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 r\cos(\varphi)\; r^2 \sin(\varphi)\; dr\, d\varphi\, d\theta$$

$$= \frac{1}{4} r^4 \Big|_0^1 \; 2\pi \; \frac{1}{2}\sin^2(\varphi)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\,\frac{1}{2} = \underline{\underline{\frac{\pi}{4}}}. \qquad 2$$

Exercice 4 5 pts

Let $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a parametric surface of class C^1, with a boundary c described by a parametric curve of class C^1. Let φ be a vector field on $\mathbb{R}^3$, of class C^1. If the orientation condition is satisfied, then

Stokes' theorem:
$$\iint_{f(\Omega)} (\operatorname{curl} \varphi) \cdot N_f \, d\sigma = \int_c \varphi \qquad 2$$

surface integral — vector normal to the surface — curve integral

In the current application: $c(t) = \begin{pmatrix} 2\sin(t) \\ 0 \\ 2\cos(t) \end{pmatrix}$ for $t \in [0, 2\pi]$

$$f : \underbrace{[0,\pi) \times [0,\pi)}_{=\Omega} \ni (\theta,\varphi) \longmapsto \begin{pmatrix} 2\cos(\theta)\sin(\varphi) \\ 2\sin(\theta)\sin(\varphi) \\ 2\cos(\theta) \end{pmatrix} \qquad 1$$

⚠

Then $\int_c \varphi = \int_0^{2\pi} \varphi(c(t)) \cdot c'(t)\, dt$

$= \int_0^{2\pi} \begin{pmatrix} 2\sin(t) \\ 2\cos(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2\cos(t) \\ 0 \\ -2\sin(t) \end{pmatrix} dt$

1

$= \int_0^{2\pi} 4 \cos(t) \sin(t)\, dt = 0.$

On the other hand: $\operatorname{curl} \varphi(x,y,z) = \begin{pmatrix} -1-1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$,

$\iint_\Omega \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \cdot 2\sin(\varphi) \begin{pmatrix} 2\cos(\theta)\sin(\varphi) \\ 2\sin(\theta)\sin(\varphi) \\ 2\cos(\theta) \end{pmatrix}$.

$= \iint_\Omega -8 \cos(\theta) \sin^2(\varphi)\, d\theta\, d\varphi$

$= -8 \sin(\theta) \Big|_0^{\pi} \int_0^{2\pi} \sin^2(\varphi)\, d\varphi$

$= 0.$ same result.

1

Exercise 5 5 pts

i) area of $f(\Omega) = \iint_\Omega \| N_f(x,y) \|\, dx\, dy$

1

with $N_f(x,y) = \partial_1 f(x,y) \times \partial_2 f(x,y)$.

1 variable 2nd variable

ii) One has $\partial_s f(\phi(s,t)) = [\partial_1 f](\phi(s,t)) [\partial_s \phi_1](s,t) + [\partial_2 f](\phi(s,t)) [\partial_s \phi_2](s,t)$

and $\partial_t f(\phi(s,t)) = [\partial_1 f](\phi(s,t)) [\partial_t \phi_1(s,t)] + [\partial_2 f](\phi(s,t)) [\partial_t \phi_2](s,t)$

⚠ they are still vectors.

Then, without writing the variables:

$$\partial_s f \times \partial_t f = {}^T\Big(\big\{[\partial_1 f_2][\partial_s \phi_1] + [\partial_2 f_2][\partial_s \phi_2]\big\}\cdot\big\{[\partial_1 f_3][\partial_t \phi_1] + [\partial_2 f_3][\partial_t \phi_2]\big\}$$
$$- \big\{[\partial_1 f_3][\partial_s \phi_1] + [\partial_2 f_3][\partial_s \phi_2]\big\}\cdot\big\{[\partial_1 f_2][\partial_t \phi_1] + [\partial_2 f_2][\partial_t \phi_2]\big\}, \dots,$$
$$\dots \Big)$$

this is for the first component of the cross product

$$= {}^T\Big(\big\{[\partial_s \phi_1][\partial_t \phi_2] - [\partial_s \phi_2][\partial_t \phi_1]\big\}[\partial_1 f_2][\partial_2 f_3]$$
$$+ \big\{[\partial_s \phi_2][\partial_t \phi_1] - [\partial_s \phi_1][\partial_t \phi_2]\big\}[\partial_1 f_3][\partial_2 f_2],$$
$$\dots, \dots \Big)$$

$$= {}^T\Big([\partial_s \phi_1][\partial_t \phi_2] - [\partial_s \phi_2][\partial_t \phi_1] \Big)\Big([\partial_1 f_2][\partial_2 f_3] - [\partial_1 f_3][\partial_2 f_2],$$
$$\dots, \dots \Big)$$

$$= \left[\det\begin{pmatrix} \partial_s \phi_1 & \partial_t \phi_1 \\ \partial_s \phi_2 & \partial_t \phi_2 \end{pmatrix}\right]\Big(\partial_1 f \wedge \partial_2 f \Big). \qquad 3$$

iii) $$\text{Area } f(\phi(D)) = \iint_D \| \partial_s f(\phi(s,t)) \times \partial_t f(\phi(s,t)) \| \, ds\, dt$$

$$= \iint_D \left| \det\begin{pmatrix} \partial_s \phi_1 & \partial_t \phi_1 \\ \partial_s \phi_2 & \partial_t \phi_2 \end{pmatrix}(s,t) \right| \, \| \partial_1 f(\phi(s,t)) \times \partial_2 f(\phi(s,t)) \| \, ds\, dt$$

by ii)

$$= \iint_D \| N_f(\phi(s,t)) \| \, |\det D\phi(s,t)| \, ds\, dt$$

change of coordinates in a volume integral

$$= \iint_\Omega \| N_f(x,y) \| \, dx\, dy.$$

as in i). The area is independent of the parametrization. 1