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7.22

a) joint pdf of $\bar{X}$ and θ:

$$f(\bar{x},\theta) = f(\bar{x}|\theta)\,\pi(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{n}} \exp\left(\frac{-(\bar{x}-\theta)^2}{2\sigma^2/n}\right) \frac{1}{\sqrt{2\pi}\,\tau} \exp\left(\frac{-(\theta-\mu)^2}{2\tau^2}\right)$$

$$= \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left(-\frac{n(\bar{x}-\theta)^2}{2\sigma^2} - \frac{(\theta-\mu)^2}{2\tau^2}\right)$$

b) Let $A = \frac{n(\bar{x}-\theta)^2}{\sigma^2} + \frac{(\theta-\mu)^2}{\tau^2}$ then $f(\bar{x},\theta) = \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{A}{2}}$

$$A = \frac{n}{\sigma^2}(\bar{x}^2 - 2\bar{x}\theta + \theta^2) + \frac{1}{\tau^2}(\theta^2 - 2\theta\mu + \mu^2) = \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\theta^2 - 2\theta\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}\right) + \left(\frac{n\bar{x}^2}{\sigma^2} + \frac{\mu^2}{\tau^2}\right)$$

$$= \frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\theta^2 - 2\theta\frac{n\bar{x}\tau^2+\mu\sigma^2}{\sigma^2\tau^2} + \left(\frac{n\bar{x}^2}{\sigma^2} + \frac{\mu^2}{\tau^2}\right)$$

$$= \frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\left[\theta^2 - 2\theta\frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2} + \left(\frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}\right)^2\right] - \frac{(n\bar{x}\tau^2+\mu\sigma^2)^2}{\sigma^2\tau^2(n\tau^2+\sigma^2)} + \frac{n\bar{x}^2}{\sigma^2} + \frac{\mu^2}{\tau^2}$$

$$= \frac{n\tau^2+\sigma^2}{\sigma^2\tau^2}\left(\theta - \frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}\right)^2 + \frac{-(n^2\bar{x}^2\tau^4 + \mu^2\sigma^4 + 2n\mu\bar{x}\tau^2\sigma^2) + n\bar{x}^2(n\tau^2+\sigma^2)\tau^2 + \mu^2(n\tau^2+\sigma^2)\sigma^2}{\sigma^2\tau^2(n\tau^2+\sigma^2)}$$

$$= \frac{\tau^2+(\sigma^2/n)}{\tau^2(\sigma^2/n)}\left(\theta - \frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}\right)^2 + \frac{(\bar{x}-\mu)^2}{\tau^2+(\sigma^2/n)}$$

$$\Rightarrow f(\bar{x},\theta) = \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-\frac{A}{2}} = \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left[\frac{-1}{2\,\frac{\tau^2\sigma^2/n}{\tau^2+(\sigma^2/n)}}\left(\theta - \frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}\right)^2\right] \exp\left[-\frac{(\bar{x}-\mu)^2}{2[(\tau^2+(\sigma^2/n)]}\right]$$

Let $\alpha = \sqrt{\frac{\tau^2\sigma^2/n}{\tau^2+(\sigma^2/n)}}$ and $\beta(\underline{x}) = \frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}$

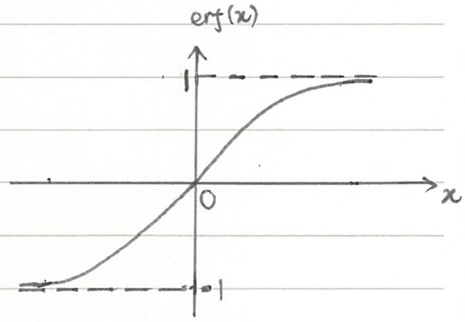
then $$f(\bar{x},\theta) = \frac{1}{\sqrt{2\pi}\,\alpha}\exp\left(\frac{-(\theta-\beta(\underline{x}))^2}{2\alpha^2}\right) \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{\tau^2+\frac{\sigma^2}{n}}}\exp\left(\frac{-(\bar{x}-\mu)^2}{2[\tau^2+(\sigma^2/n)]}\right)}_{= n[\mu;\,\tau^2+(\sigma^2/n)]} \quad (1)$$

Marginal distribution of $\bar{X}$:

$$m(\bar{x}|\sigma^2,\mu,\tau^2) = \int_{-\infty}^{\infty} f(\bar{x},\theta)\,d\theta = n\left[\mu,\tau^2+\frac{\sigma^2}{n}\right]\frac{1}{\sqrt{2\pi}\,\alpha}\int_{-\infty}^{\infty}\exp\left(\frac{-(\theta-\beta(\bar{x}))^2}{2\alpha^2}\right)d\theta$$

Let $\frac{\theta-\beta(\underline{x})}{\sqrt{2}\,\alpha} = u$ then $\int_{-\infty}^{\infty}\exp\left(\frac{-(\theta-\beta(\underline{x}))^2}{2\alpha^2}\right)d\theta = \sqrt{2}\,\alpha\int_{-\infty}^{\infty} e^{-u^2}\,du$

We use error function $\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\,dt$



Moreover, $\lim_{x\to\infty}\mathrm{erf}(x) = 1 \Rightarrow \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$

$$\Rightarrow m(\bar{x}|\sigma^2,\mu,\tau^2) = n\left[\mu,\tau^2+\frac{\sigma^2}{n}\right]\frac{1}{\sqrt{2\pi}\,\alpha}(\sqrt{2}\,\alpha\sqrt{\pi}) = n\left[\mu,\tau^2+\frac{\sigma^2}{n}\right] \quad \text{(Q.E.D)} \quad (2)$$

c) Based on (1) and (2), we know that posterior distribution of θ is:

$$\pi(\theta, \bar{x}, \sigma^2, \mu, \tau^2) = \frac{1}{\sqrt{2\pi}\,\alpha} \exp\left(\frac{-(\theta - \beta(\underline{x}))^2}{2\alpha^2}\right) = n\left[\beta(\underline{x}), \alpha\right]$$

Mean: $\beta(\underline{x}) = \dfrac{n\tau^2\bar{x} + \sigma^2\mu}{n\sigma^2 + \tau^2} = \dfrac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}}\,\bar{x} + \dfrac{\sigma^2/n}{\frac{\sigma^2}{n} + \tau^2}\,\mu$

Variance $\alpha^2 = \dfrac{\frac{\sigma^2}{n}\tau^2}{\frac{\sigma^2}{n} + \tau^2}$