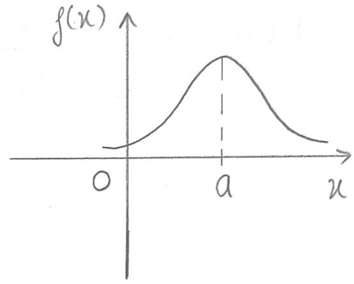
Ex 2.28, [CB]

Song Ha, Arata

a) Firstly let us calculate the mean, given that the pdf is symmetric about a point $a$. One has:




For example, a graph of $f(x)$ may look like this

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} (x+a) f(x+a)\,d(x+a)$$

$$\Leftrightarrow \mu = \underbrace{\int_{-\infty}^{\infty} x f(x+a)\,dx}_{0} + a \underbrace{\int_{-\infty}^{\infty} f(x+a)\,d(x+a)}_{1 \text{ (by definition of a pdf)}}$$

(because $x$ is odd function about 0 and $f(x+a)$ is even function about 0)

So $\mu = a$

The $n^{th}$ central moment is then: $\mu_n = \int_{-\infty}^{\infty} (x-a)^n f(x)\,dx$

$$\mu_3 = \int_{-\infty}^{\infty} (x-a)^3 f(x)\,dx = \int_{-\infty}^{\infty} y^3 f(y+a)\,dy = 0$$

(again because $y^3$ is odd and $f(y+a)$ is even)

Hence $\alpha_3 := \dfrac{\mu_3}{(\mu_2)^{3/2}} = 0$ for a symmetric pdf

$\alpha_3$ is called the skewness, which measures the lack of symmetry of the pdf. So it is not surprising that $\alpha_3 = 0$ when the pdf is symmetric about a point.

b) For example let us consider $f(x) = e^{-x}$, $x \geq 0$

(Note that $f(x)$ is not symmetric here)

Firstly, let us calculate the mean:

$$\mu = \int_0^{\infty} x e^{-x}\,dx = -xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,dx$$

$$= (0-0) + (-e^{-x})\Big|_0^{\infty} = 1$$

($\lim_{x\to\infty} x^n e^{-x} = 0$, $n \in \mathbb{N}$ using L'Hospital rule $n$ times)

$$\text{So } \mu_2 = \int_0^{\infty} (x-1)^2 e^{-x}\,dx = -(x-1)^2 e^{-x}\Big|_0^{\infty} + \int_0^{\infty} 2(x-1)e^{-x}\,dx$$

$$= 1 - 2(x-1)e^{-x}\Big|_0^{\infty} + \int_0^{\infty} 2e^{-x}\,dx$$

$$= 1 - 2 + 2 = 1$$

$$\mu_3 = \int_0^\infty (x-1)^3 e^{-x}\, dx = -(x-1)^3 e^{-x}\Big|_0^\infty + \int_0^\infty 3(x-1)^2 e^{-x}\, dx$$

$$= -1 + 3\mu_2 = -1 + 3 = 2$$

Hence $\alpha_3 = \dfrac{\mu_3}{(\mu_2)^{3/2}} = 2$

Exercise 2.28 (c1) Part (c) Calculate $\alpha_4$ Liyang, Sun Chang

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

This is the pdf of standard normal distribution.

$\Rightarrow \mu = 0, \ \mu_2 = \sigma^2 = 1$

$$\mu_4 = \int_{-\infty}^{\infty} (x-\mu)^4 f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}}\,dx$$

$$\sqrt{2\pi}\,\mu_4 = \int_{-\infty}^{\infty} x^3 x e^{-\frac{x^2}{2}}\,dx = -x^3 e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} 3x^2\,dx$$

$$= 3\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\,dx = 3\sqrt{2\pi}\,\mu_2 = 3\sqrt{2\pi}$$

$\Rightarrow \mu_4 = 3$

$\Rightarrow \alpha_4 = \frac{\mu_4}{\mu_2^2} = 3$ $\left(\alpha_4 := \frac{\mu_4}{\mu_2^2}\right.$ is called kurtosis which measures the peakedness or flatness of the pdf$\left.\right)$

(c3)

$f(x) = e^{-|x|}$

$$\mu_2 = \int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx = 2\int_0^{\infty} x^2 e^{-x}\,dx = 2\left(-x^2 e^{-x}\Big|_0^{\infty} + \int_0^{\infty} 2x e^{-x}\,dx\right) = 4\left(-x e^{-x}\Big|_0^{\infty} + \int e^{-x}\,dx\right) = 4.$$

$$\mu_4 = \int_{-\infty}^{\infty} x^4 e^{-|x|}\,dx = 2\int_0^{\infty} x^4 e^{-x}\,dx = 2\left(-x^4 e^{-x}\Big|_0^{\infty} + 4\int x^3 e^{-x}\,dx\right) = 8\left(-x^3 e^{-x} + \int_0^{\infty} 3x^2 e^{-x}\,dx\right)$$

$$= 12 \cdot 2\int_0^{\infty} x^2 e^{-x}\,dx = 12\mu_2 = 48$$

$\Rightarrow \alpha_4 = \mu_4/\mu_2^2 = 48/4^2 = 48/16 = 3.$

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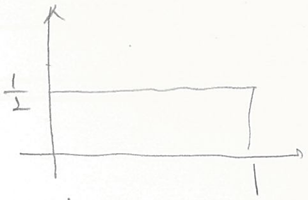
$f(x) = \frac{1}{2} \quad , -1 < x < 1$




$\underbrace{1, 2, 3, 4, 5, 6}_{\frac{1}{6}}$

$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}$

( Part (c) calculate $\alpha_4$ for the given pdf )

$$\mu = \int_{-1}^{1} \frac{x}{2} \, dx$$

$$= \left[ \frac{x^2}{4} \right]_{-1}^{1}$$

$$= 0$$

$$\mu_2 = \int_{-1}^{1} x^2 \cdot \frac{1}{2} \, dx$$

$$= \left[ \frac{1}{6} x^3 \right]_{-1}^{1}$$

$$= \frac{1}{3}$$

$$\mu_3 = \int_{-1}^{1} x^3 \cdot \frac{1}{2} \, dx$$

$$= \left[ \frac{x^4}{8} \right]_{-1}^{1}$$

$$= 0$$

$$\mu_4 = \int_{-1}^{1} x^4 \cdot \frac{1}{2} \, dx$$

$$= \left[ \frac{x^5}{10} \right]_{-1}^{1}$$

$$= \frac{1}{5}$$

$$\alpha_4 = \frac{\frac{1}{5}}{\frac{1}{9}} = \frac{9}{5}$$