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4.2. CONDITIONAL DISTRIBUTIONS AND INDEPENDENCE

4.2.1. DEFINITION

Let (X,Y) be a discrete bivariate random vector with joint pmf $f(x,y)$ and marginal pmfs $f_X(x)$ and $f_Y(y)$. For any x such that $P(X=x) = f_X(x) > 0$, the conditional pmf of Y given that $X = x$ is the function of y denoted by $f(y|x)$ and defined by

$$f(y|x) = P(Y=y \mid X=x) = \frac{f(x,y)}{f_X(x)}$$

For any y such that $P(Y=y) = f_Y(y) > 0$, the conditional pmf of X given that $Y=y$ is the function of x denoted by $f(x|y)$ and defined by

$$f(x|y) = P(X=x \mid Y=y) = \frac{f(x,y)}{f_Y(y)}$$

EXAMPLE (4.2.2)

Let us define the joint pmf of (X,Y) by

$$f(0,10) = f(0,20) = \frac{2}{18}$$
$$f(1,10) = f(1,30) = \frac{3}{18}$$
$$f(1,20) = f(2,30) = \frac{4}{18}$$

For each of the possible values of X, $x = 0,1,2$.

The marginal pmf of X is
$$f_X(0) = f(0,10) + f(0,20) = \frac{4}{18}$$
$$f_X(1) = f(1,10) + f(1,20) + f(1,30) = \frac{10}{18}$$
$$f_X(2) = f(2,30) = \frac{4}{18}$$

For $x = 0$, $f(0,y)$ is positive for $y=10$ and $y=20$.
Thus, $f(y|0)$ is positive for $y=10$ and $y=20$, and

$$f(10|0) = \frac{f(0,10)}{f_X(0)} = \frac{1}{2}$$

$$f(20|0) = \frac{f(0,20)}{f_X(0)} = \frac{1}{2}$$

So, given that $X=0$, the conditional probability distribution for Y is discrete, and assigns probability $\frac{1}{2}$ to each point $y=10$ and $y=20$.

For $x = 1$, $f(y|1)$ is positive for $y = 10, 20, 30$, and

$$f(10|1) = f(30|1) = \frac{3}{10}$$

$$f(20|1) = \frac{4}{10}$$

For $x = 2$, $f(30|2) = 1$

→ for this, if we know that $X = 2$, then we know that $Y = 30$.

4.2.3. DEFINITION

Let (X,Y) be a continuous bivariate random vector with joint pdf $f(x,y)$ and marginal pdfs $f_X(x)$ and $f_Y(y)$. For any x such that $f_X(x) > 0$, the conditional pdf of Y given that $X = x$ is the function of y denoted by $f(y|x)$ and defined by

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

For any y such that $f_Y(y) > 0$, the conditional pdf of X given that $Y=y$ is the function of x denoted by $f(x|y)$ and defined by

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

EXAMPLE (4.2.4)

Let the continuous random vector (X, Y) have the joint pdf $f(x,y) = e^{-y}$, $0 < x < y < \infty$.
Let us compute the conditional pdf of Y given $X = x$.
If $x \leq 0$, $f(x,y) = 0$ for all y, so $f_X(x) = 0$.
If $x > 0$, $f(x,y) > 0$ if $y > x$.

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} e^{-y}\,dy = e^{-x}$$

Thus, X has an exponential distribution.
For any $x > 0$ ($\Rightarrow f_X(x) > 0$),

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \quad \text{if } y > x$$

$$\text{and } f(x \mid y) = \frac{f(x,y)}{f_X(x)} = \frac{0}{e^{-x}} = 0 \quad \text{if } y \leq x$$

4.2.5. DEFINITION

Let (X, Y) be a bivariate random vector with joint pdf or pmf $f(x,y)$ and marginal pdfs or pmfs $f_X(x)$ and $f_Y(y)$. Then X and Y are called <u>independent random variables</u> if, for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$f(x,y) = f_X(x) f_Y(y)$$

If X and Y are independent, the conditional pdf of Y given $X = x$ is

$$\begin{aligned} f(y \mid x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{f_X(x) f_Y(y)}{f_X(x)} \\ &= f_Y(y) \quad \text{(which is independent of } x\text{)} \end{aligned}$$

4.2.10. THEOREM

Let X and Y be independent random variables

a. For any $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$,

$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$

(the events $\{X \in A\}$ and $\{Y \in B\}$ are independent events)

b. Let $g(x)$ be a function of x, and $h(y)$ be a function of y.

Then $E(g(X)h(Y)) = E(g(X))\,E(h(Y))$

PROOF

$$\begin{aligned} E(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_X(x) f_Y(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \int_{-\infty}^{\infty} h(y) f_Y(y)\,dy \\ &= E(g(X))\,E(h(Y)) \end{aligned}$$

$\square$

EXAMPLE (4.2.11)

Let X and Y be independent exponential random variables. From the theorem above, we have

$$P(X \geq 4, Y < 3) = P(X \geq 4) P(Y < 3) = e^{-4}(1 - e^{-3})$$

Let $g(x) = x^2$ and $h(y) = y$, then

$$\begin{aligned} E(X^2 Y) &= E(X^2) E(Y) \\ &= (\mathrm{Var}(X) + E(X)^2) E(Y) = \\ &= (1 + 1^2) 1 = 2 \end{aligned}$$

4.2.12 THEOREM

Let X and Y be independent random variables with moment generating function $M_X(t)$ and $M_Y(t)$. Then the moment generating function of the random variable $Z = X + Y$ is given by

$$M_Z(t) = M_X(t)\, M_Y(t)$$

PROOF
$$\begin{aligned} M_Z(t) &= E(e^{tZ}) \\ &= E(e^{t(X+Y)}) \\ &= E(e^{tX} e^{tY}) \\ &= E(e^{tX})\, E(e^{tY}) \\ &= M_X(t)\, M_Y(t) \end{aligned}$$ $\square$

EXERCISES

4.10. The random pair (X, Y) has the distribution

		X = 1	X = 2	X = 3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

a. Show that X and Y are dependent

X and Y are independent if $f(x,y) = f_X(x) f_Y(y)$

For $x = 2$, $f_X(2) = f(2,2) + f(2,3) + f(2,4)$

$$= \frac{1}{6} + 0 + \frac{1}{3}$$
$$= \frac{1}{2}$$

For $y = 3$, $f_Y(3) = f(1,3) + f(2,3) + f(3,3)$

$$= \frac{1}{6} + 0 + \frac{1}{6}$$
$$= \frac{1}{3}$$

So, $f(2,3) = 0 \neq \frac{1}{6} = f_X(2) f_Y(3)$

$\Rightarrow$ X and Y are not independent, so they are dependent

b. Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Let $f_X(x) = f_U(u)$ and $f_Y(y) = f_V(v)$

For $u = 1$, $f_U(1) = \frac{1}{4}$ For $v = 2$, $f_V(2) = \frac{1}{3}$

$u = 2$, $f_U(2) = \frac{1}{2}$ $v = 3$, $f_V(3) = \frac{1}{3}$

$u = 3$, $f_U(3) = \frac{1}{4}$ $v = 4$, $f_V(4) = \frac{1}{3}$

As U and V are independent, $f(u,v) = f_U(u) f_V(v)$.

This generates the probability table:

		U = 1	U = 2	U = 3
V	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

4.11. Let U = the number of trials needed to get the first head and V = the number of trials needed to get two heads in repeated tosses of a fair coin. Are U and V independent variables?

If we know $U = u$, then we know $V > u$.

$f(u, v) = f(U = u, V = v) = \left(\frac{1}{2}\right)^v = f_V(v) \neq f_U(u) f_V(v)$

As $f(u, v) \neq f_U(u) f_V(v)$, U and V are not independent variables.

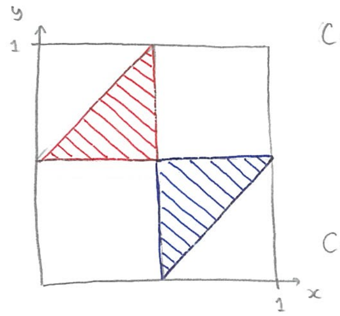
4.12 If a stick is broken at random into three pieces, what is the probability that the pieces can be put together in a triangle?

Let X and Y be the two points where the stick is broken.

$P(X = x)$ and $P(Y = y)$ a continuous uniform distribution function, and are independent.

Suppose the length of the stick is 1. Then, the joint distribution of X and Y is uniform on the unit square.

By the triangle inequality, the length of each piece must be smaller than $\frac{1}{2}$ in order to get three pieces that can form a triangle. (1)



Case 1 : $x < y$

$(1) \Leftrightarrow \begin{cases} x < \frac{1}{2} \\ y - x < \frac{1}{2} \\ 1 - y < \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x < \frac{1}{2} \\ y < x + \frac{1}{2} \\ y > \frac{1}{2} \end{cases}$ $\Leftrightarrow (x, y)$ belongs to the triangle defined by the three vertices $(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$

(shown in red)

Case 2 : $x > y$

$(1) \Leftrightarrow \begin{cases} y < \frac{1}{2} \\ x - y < \frac{1}{2} \\ 1 - x < \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y < \frac{1}{2} \\ x < y + \frac{1}{2} \\ x > \frac{1}{2} \end{cases}$ $\Leftrightarrow (x, y)$ belongs to the triangle defined by the three vertices $(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2})$

(shown in blue)

As the joint distribution of X and Y is uniform on the unit square, the probability that (X, Y) belongs to each triangle is $\frac{1}{8}$. Thus, the probability that the pieces can form a triangle is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.