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< The third report >

Exercise 4.1.1

Such a function F is monotone non-decreasing and right continuous, i.e. $F(x) = F(x+0) = \lim_{\lambda \rightarrow x^+} F(\lambda+0)$ for all $x \in \mathbb{R}$.

If the function F is discontinuous at $x \in \mathbb{R}$

then $F(N) - F(x-0) > 0$.

Set $D_n = \{x \in \mathbb{R} \mid F(x) - F(x-0) > \frac{1}{n}\}$, and let D be the set of discontinuity points of F ,

then $D = \bigcup_{n \in \mathbb{N}} D_n$.

Assume D is uncountable,

in this case, one of D_n should be uncountable.

Since F is a monotone and bounded function,

then F is bounded on any bounded interval

thus for any M , $D_n \cap [-M, M]$ is finite.

such that $D_n = \bigcup_{M \in \mathbb{N}} (D_n \cap [-M, M])$ is countable.

which is contradict to the assumption

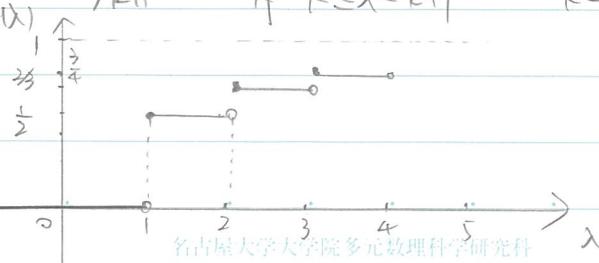
i.e. such a function F has at most a countable set of points of discontinuity

Exercise 4.1.2.

(i) Example 4.1:

F is a piece-wise constant function.

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{k}{k+1} & \text{if } k \leq x < k+1 \end{cases} \quad k=1, 2, \dots$$



since if $\lambda \notin \{1, 2, 3, \dots\} = N$ then $m_F(\{\lambda\}) = 0$

$$\text{if } \lambda \in \{1, 2, 3, \dots\} = N \text{ then } m_F(\{\lambda\}) = \frac{\lambda}{\lambda+1} - \frac{\lambda-1}{\lambda} = \frac{1}{\lambda(\lambda+1)}$$

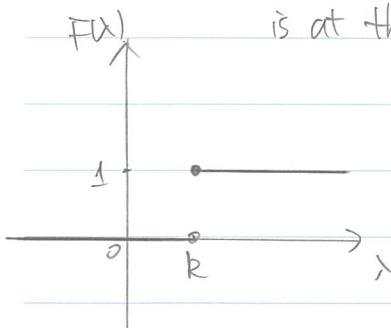
so for any Borel set V , one has,

$$m_F(V) = \sum_{\lambda \in V} m_F(\{\lambda\}) = \sum_{\lambda \in V \cap N} \frac{1}{\lambda(\lambda+1)}$$

(ii) Example 4.2:

the function F assumes only the values 0 and 1, and the jump of F

is at the point $k \in \mathbb{R}$.



$$F(\lambda) = \begin{cases} 0 & \lambda < k \\ 1 & \lambda \geq k \end{cases}$$

so for any Borel set V ,

$$\text{if } k \in V \text{ then } m_F(V) = 1 - 0 = 1$$

$$\text{if } k \notin V \text{ then } m_F(V) = 1 - 1 = 0 \text{ or } m_F(V) = 0 - 0 = 0$$

thus the Stieltjes measure is $m_F(V) = \begin{cases} 1 & k \in V \\ 0 & k \notin V \end{cases}$
(dirac measure)

(iii) Example 4.3:

The jump function : if $\lambda \leq 0$ $F(\lambda) = 0$

if $\lambda \geq 1$ $F(\lambda) = 1$

$$\text{if } \lambda \in (0, 1) \text{ } F(\lambda) = \sum_{r_j \leq \lambda} 2^{-j}$$

$\{r_j\}$ is an enumeration of the rational numbers in $(0, 1)$

F is continuous except at rational points $\{r_j\}$.

For any Borel set V ,

$$m_F(V) = \sum_{r_j \in V} m_F(\{r_j\}) = \sum_{r_j \in V} 2^{-j} \text{ is the Stieltjes measure,}$$

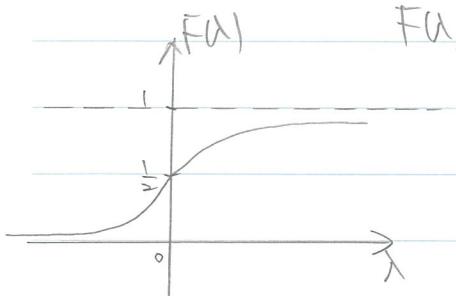
since the measure on other parts are zero.

(pure point measure)

(iv) Example 4.4 :

The function F is a continuous function for example

$$F(\lambda) = \frac{1}{2} + \frac{1}{\pi} \arctan(\lambda), \quad F(\lambda) \text{ is absolutely continuous.}$$



for each point $\lambda \in \mathbb{R}$, $m_F(\{\lambda\}) \leq F(\lambda) - F(\lambda - \omega) = 0$.

so $m_F(\{\lambda\}) = 0$ for each point of \mathbb{R}

In this case, for any Borel set V , the measure $m_F(V)$ can be obtained by integrating the derivative of F over V ,

$$F'(\lambda) = \frac{1}{\pi} \cdot \frac{1}{1+\lambda^2}$$

$$\text{so one has } m_F(V) = \int_{-\infty}^{\infty} F'(\lambda) \chi_V(\lambda) d\lambda = \frac{1}{\pi} \int_V \frac{1}{1+\lambda^2} d\lambda$$

(v) Example 4.5 :

(singular continuous)

- The function F is the Cantor function

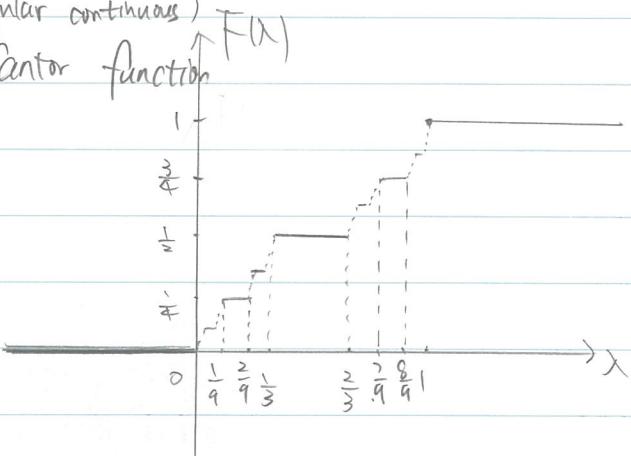
$$\text{for } \lambda \leq 0 \quad F(\lambda) = 0.$$

$$\lambda \geq 1 \quad F(\lambda) = 1$$

$$\lambda \in (\frac{1}{3}, \frac{2}{3}) \quad F(\lambda) = \frac{1}{2}$$

$$\lambda \in (\frac{1}{9}, \frac{2}{9}) \quad F(\lambda) = \frac{1}{4}$$

$$\lambda \in (\frac{7}{9}, \frac{8}{9}) \quad F(\lambda) = \frac{3}{4} \dots$$



$$(0, \frac{1}{3}), (\frac{1}{3}, 1), (0, \frac{2}{3}), (\frac{2}{3}, 1), (\frac{1}{9}, \frac{1}{3}), (\frac{2}{9}, \frac{2}{3}), (\frac{7}{9}, \frac{8}{9}) \dots$$

these small intervals are called Cantor sets, C^n

- The Cantor function is a non-decreasing continuous function which is differentiable at each point in \mathbb{R} not belonging to the Cantor sets hence almost everywhere w.r.t. the Lebesgue measure and the derivative is 0
- For any Borel set V , if $V \cap C^n = \emptyset$, then $m_F(V) = 0$. for $\lambda \notin C^n$
- the measure is supported on the Cantor set, which is a Borel set of Lebesgue measure zero