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< The Second Report >

Exercise 2.1.13.

Let A be a symmetric operator.

Suppose $\{\varphi_n\}_{n \in \mathbb{N}}$ is a sequence in $D(A^*)$ and $S - \lim_{n \rightarrow \infty} \varphi_n = \varphi \in \mathcal{H}$.

Suppose that $S - \lim_{n \rightarrow \infty} A^* \varphi_n = \psi \in \mathcal{H}$.

For any $\psi \in D(A)$, one has

$$\langle \varphi, A\psi \rangle = \lim_{n \rightarrow \infty} \langle \varphi_n, A\psi \rangle = \lim_{n \rightarrow \infty} \langle A^* \varphi_n, \psi \rangle = \langle \varphi, \psi \rangle.$$

which shows that $\langle \varphi, A\psi \rangle = \langle A^*\varphi, \psi \rangle = \langle \varphi, \psi \rangle$.

such that $\varphi \in D(A^*)$ and $A^*\varphi = \psi$, thus A^* is closed.

Since A is a symmetric operator, then for $f, g \in D(A)$, $\langle Af, g \rangle = \langle f, Ag \rangle$.

then for $f \in D(A)$, there exists a vector f^* with $Af = f^* \in \mathcal{H}$.

such that $\langle f, Ag \rangle = \langle f^*, g \rangle$ for all $g \in D(A)$.

Since $\langle f^*, g \rangle = \langle f, Ag \rangle = \langle A^*f, g \rangle$ i.e. $A^*f = f^*$.

then one has $A^*f = Af$ for all $f \in D(A)$.

so $(A^*, D(A^*))$ is an extension of $(A, D(A))$.

thus A has a closed extension and therefore A is closable.

Exercise 2.3.2

(i). Since A is self-adjoint, one has the relation that for $z = \lambda + i\mu$, $\lambda \in \mathbb{R}$,

$$\begin{aligned} \| (A-z)f \|^2 &= \| (A-\lambda)f \|^2 + \mu^2 \| f \|^2 - i\mu \langle (A-\lambda)f, f \rangle + i\mu \langle f, (A-\lambda)f \rangle \\ &= \| (A-\lambda)f \|^2 + \mu^2 \| f \|^2. \end{aligned}$$

$$\text{so for } \| Bf \|^2 \leq a^2 \| Af \|^2 + b^2 \| f \|^2 = a^2 \left[\| Af \|^2 + \frac{b^2}{a^2} \| f \|^2 \right],$$

$$\text{it is equivalent to } \| Bf \|^2 \leq a^2 \| (A \pm i\frac{b}{a})f \|^2. \quad \forall f \in D(A).$$

Since $\pm i\frac{b}{a} \in P(A)$ and $(A \pm i\frac{b}{a})^{-1}$ gives a bijection between \mathcal{H} and $D(A)$,

thus $\| Bf \|^2 \leq a^2 \| (A \pm i\frac{b}{a})f \|^2$ is equivalent to

$$\| B(A \pm i\frac{b}{a})^{-1}g \|^2 \leq a^2 \| g \|^2 \quad \forall g \in \mathcal{H}.$$

Since $a > 0$,

$$\text{such that } \| B(A \pm i\frac{b}{a})^{-1}g \|^2 \leq a \| g \|^2. \quad \text{i.e. } \| B(A \pm i\frac{b}{a})^{-1} \|^2 \leq a.$$

(ii). Set $\alpha' = \|B(A-\bar{z})^{-1}\|$ and for $\forall f \in D(A)$,

$$\begin{aligned} \text{one has } \|Bf\| &= \|B(A-\bar{z})^{-1}(A-\bar{z})f\| \\ &\leq \|B(A-\bar{z})^{-1}\| \cdot \|(A-\bar{z})f\| \\ &\leq \alpha' [\|Af\| + |\bar{z}| \|f\|] \\ &= \alpha' \|Af\| + \alpha' |\bar{z}| \|f\| \end{aligned}$$

which means that B is A -bounded.

(iii). By (ii), we have obtained that there exists $k = \frac{b}{a} > 0$
s.t. $B(A+ik)^{-1} \in (\mathcal{B}(\mathcal{H}))^*$

If $z \in \rho(A)$, then by the first resolvent equation:

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2) (A-z_1)^{-1} (A-z_2)^{-1}$$

$$\text{we can get } B(A-z)^{-1} - B(A-ik)^{-1} = (z-ik) B(A-ik)^{-1} (A-z)^{-1}$$

$$\text{and then } B(A-z)^{-1} = B(A-ik)^{-1} + (z-ik) [B(A-ik)^{-1}] (A-z)^{-1}$$

Since the RHS is bounded, so $B(A-z)^{-1} \in (\mathcal{B}(\mathcal{H}))^*$

Then let α be the A -bound of B , by (iii), we know that
for $\alpha' = \|B(A-\bar{z})^{-1}\|$, there has $\|Bf\| \leq \alpha' [\|Af\| + |\bar{z}| \|f\|]$

$$\text{so } \alpha \leq \inf_{z \in \rho(A)} \alpha' = \inf_{z \in \rho(A)} \|B(A-z)^{-1}\|$$

Since for any $\alpha' > \alpha$, $\|Bf\| \leq \alpha' \|Af\| + \beta \|f\|$ with $\beta = |\bar{z}| \cdot \alpha'$

Then for $\forall \varepsilon > 0$, it's also true for $\alpha' = \alpha + \frac{\varepsilon}{2}$

For $\delta, \eta \in \mathbb{R}$ and $\delta > 0$, since $0 \leq (\delta^{\frac{1}{2}} \delta - \delta^{-\frac{1}{2}} \eta)^2 \leq 8\delta^2 - 2\delta\eta + \delta^{-1}\eta^2$

$$\begin{aligned} \text{so } (\delta+\eta)^2 &= \delta^2 + 2\delta\eta + \eta^2 \leq \delta^2 + 8\delta^2 + \delta^{-1}\eta^2 + \eta^2 \\ &= (1+8)\delta^2 + (1+\delta^{-1})\eta^2 \end{aligned}$$

let $\delta = \alpha' \|Af\|$ and $\eta = \beta' \|f\|$ and since B is A -bounded,

$$\text{then } \|Bf\|^2 \leq [\alpha' \|Af\| + \beta' \|f\|]^2 \leq (1+8)\alpha'^2 \|Af\|^2 + (1+\delta^{-1})\beta'^2 \|f\|^2$$

Take δ sufficiently small, we can get $\|Bf\|^2 \leq \alpha'^2 \|Af\|^2 + b^2 \|f\|^2$ is

true for all $a > \alpha'$

Then let $\alpha = \omega + \frac{b}{2} = \omega + \frac{\omega}{2} + \frac{b}{2} = \omega + b$ and $b > 0$

by (i) one has
for $\forall w > 0$, $\|B(A + ik)^{-1}\| \leq \omega + w$ with $k = \frac{b}{\omega + w}$.

i.e. $\omega = \inf_{z \in \rho(A)} \|B(A - z)^{-1}\| = \inf_{z \in \rho(A)} \|B(A \pm ik)^{-1}\|$